Exercise 8.1

(a) Let $(X, \|\cdot\|_X)$ be a separable normed $\mathbb{K}$-vector space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). Prove that the weak* topology on the unit ball $B^* := \{\varphi \in X^* : \|\varphi\|_{X^*} \leq 1\}$ of X^* is metrizable.

(b) If X^* is separable, then (B, τ_{w}) is metrizable.

Note: (X, τ_{w}) is **not** metrizable when $\dim X = \infty$, as we saw on the last problem set!

Exercise 8.2

(a) Let $(X, \|\cdot\|_X)$ be a normed space and let $\emptyset \neq Q \subset X$ be an open, convex subset containing the origin. Prove that there exists a subset $\Upsilon \subset X^*$ such that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\},$$

which means that Q is an intersection of open, affine half-spaces.

(b) *Definition.* Let $(X, \|\cdot\|_X)$ be a normed space. The convex hull of $A \subset X$ is defined as

$$\operatorname{conv}(A) := \bigcap_{B \supset A, B \text{ convex}} B$$

Recall the following representation theorem for convex hulls

$$\operatorname{conv}(A) = \left\{ \sum_{k=1}^{n} \lambda_k x_k \mid n \in \mathbb{N}, x_1, \ldots, x_n \in A, \lambda_1, \ldots, \lambda_n \geq 0, \sum_{k=1}^{n} \lambda_k = 1 \right\}.$$

Using the representation of the convex hull above, prove that if $(X, \|\cdot\|_X)$ is a normed space and $A, B \subset X$ are compact, convex subsets, then $\operatorname{conv}(A \cup B)$ is compact.

Exercise 8.3 Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Given a positive integer n, consider n pairwise distinct points $x_1, \ldots, x_n$ in X and the functional

$$F : X \to \mathbb{R}, \quad F(x) = \sum_{i=1}^{n} \|x - x_i\|_X^2$$

(a) Prove that the functional F has a global minimum on X, namely the value $\inf_{x \in X} F(x)$ is a real number attained by F at some $\bar{x} \in X$.

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(b) Let us now assume that $(X, \|\cdot\|_X)$ is a Hilbert space (thus $\|\cdot\|_X$ is induced by a scalar product $\langle\cdot,\cdot\rangle_X$). Prove that the minimum $\bar{x} \in X$ is unique, and that $\bar{x}$ belongs to the convex hull K of $\{x_1, \ldots, x_n\}$.

Exercise 8.4 Let $m \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^m$ be a bounded measurable set with $|\Omega| > 0$. For $g \in L^2(\mathbb{R}^m, \mathbb{R})$, we define the map

$$\begin{aligned} V : L^2(\Omega, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \int_\Omega \int_\Omega g(x-y) f(x) f(y)\, dy\, dx \end{aligned}$$

and given $h \in L^2(\Omega, \mathbb{R})$, we define the map

$$\begin{aligned} E : L^2(\Omega, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \|f-h\|^2_{L^2(\Omega,\mathbb{R})} + V(f). \end{aligned}$$

(a) Prove that V is weakly sequentially continuous by proceeding as follows.

(i) Show that the linear operator $T : L^2(\Omega, \mathbb{R}) \to L^2(\Omega, \mathbb{R})$ mapping $f \mapsto Tf$ given by

$$(Tf)(x) = \int_\Omega g(x-y) f(y)\, dy$$

is well-defined.

(ii) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega, \mathbb{R})$ such that $f_k \xrightarrow{w} f$ in $L^2(\Omega, \mathbb{R})$ as $k \to \infty$. Prove that $\|Tf_k - Tf\|_{L^2(\Omega,\mathbb{R})} \to 0$ as $k \to \infty$, where T is as in (i).

(iii) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega, \mathbb{R})$ such that $f_k \xrightarrow{w} f$ in $L^2(\Omega, \mathbb{R})$ as $k \to \infty$. Show that $V(f_k) \to V(f)$ as $k \to \infty$, i. e. V is weakly sequentially continuous.

(b) Under the assumption $g \geq 0$ almost everywhere, prove that E restricted to

$$L^2_+(\Omega, \mathbb{R}) := \left\{ f \in L^2(\Omega, \mathbb{R}) \mid f(x) \geq 0 \text{ for almost every } x \in \Omega \right\}$$

attains a global minimum.