Exercise 8.1

(a) Let $(X, \|\cdot\|_X)$ be a separable normed $\mathbb{K}$-vector space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). Prove that the weak* topology on the unit ball $B^* := \{\varphi \in X^* : \|\varphi\|_{X^*} \leq 1\}$ of X^* is metrizable.

(b) If X^* is separable, then (B, τ_{w}) is metrizable.

Note: (X, τ_{w}) is **not** metrizable when $\dim X = \infty$, as we saw on the last problem set!

Solution.

(a) Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a dense subset of the unit ball $B := \{x \in X : \|x\|_X \leq 1\}$ in X. The fact that $\sup_{n\in\mathbb{N}} \|x_n\|_X \leq 1$ ensures that the mapping $d : B^* \times B^* \to [0, \infty)$, given by

$$d(\varphi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \left|\varphi\left(x_n\right) - \psi\left(x_n\right)\right| \quad \text{for all } \varphi, \psi \in B^*,$$

is well-defined. Indeed:

$$\begin{aligned} 0 \leq \sum_{n=1}^{\infty} 2^{-n} \left|\varphi\left(x_n\right) - \psi\left(x_n\right)\right| &\leq \sum_{n=1}^{\infty} 2^{-n} \|\varphi - \psi\|_{X^*} \|x_n\|_X \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \|\varphi - \psi\|_{X^*} \leq \|\varphi - \psi\|_{X^*} \quad \text{for all } \varphi, \psi \in B^*. \end{aligned}$$

We claim that d is a metric on B^*. For this, note that symmetry is clear. Moreover, for all $\varphi, \psi, \xi \in B^*$, we obtain

$$\begin{aligned} d(\varphi, \xi) &= \sum_{n=1}^{\infty} 2^{-n} \left|\varphi\left(x_n\right) - \xi\left(x_n\right)\right| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left|\varphi\left(x_n\right) - \psi\left(x_n\right)\right| + \sum_{n=1}^{\infty} 2^{-n} \left|\psi\left(x_n\right) - \xi\left(x_n\right)\right| \\ &= d(\varphi, \psi) + d(\psi, \xi), \end{aligned}$$

that is, the triangle inequality holds. Finally, for $\varphi, \psi \in B^*$ we can infer from $d(\varphi, \psi) = 0$ that $\varphi\left(x_n\right) = \psi\left(x_n\right)$ for all $n \in \mathbb{N}$. Hence, any $\varphi, \psi \in B^*$ with $d(\varphi, \psi) = 0$ have to coincide on $\operatorname{span}\{x_n \mid n \in \mathbb{N}\}$ because of linearity and even on $\overline{\operatorname{span}\{x_n \mid n \in \mathbb{N}\}}$ because of continuity. As $\overline{\operatorname{span}\{x_n \mid n \in \mathbb{N}\}} = X$ due to $(x_n)_{n\in\mathbb{N}}$

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lying dense in the unit ball B of X, we obtain that any $\varphi, \psi \in B^*$ with $d(\varphi, \psi) = 0$ have to be identical.

All of the above is useless if we cannot show that the weak* topology τ_{w^*} on B^* is equal to the topology τ_{d} on B^* which is induced by the metric d. Next, we are going to show that $\tau_{\mathrm{d}} \subseteq \tau_{\mathrm{w}^*}$ and $\tau_{\mathrm{w}^*} \subseteq \tau_{\mathrm{d}}$.

"$\tau_{\mathrm{d}} \subseteq \tau_{\mathrm{w}^*}$" Let $O \in \tau_{\mathrm{d}}$ and $\varphi \in O$ be arbitrary. Then there exists $\varepsilon \in (0, \infty)$ such that $\{\psi \in B^* \mid d(\varphi, \psi) < \varepsilon\} \subseteq O$. With $N \in \mathbb{N}$ so that $2^{-N} < \frac{\varepsilon}{4}$, we get that

$$\begin{aligned}\sum_{n=N+1}^{\infty} 2^{-n} \left|\varphi\left(x_n\right) - \psi\left(x_n\right)\right| &\leq \sum_{n=N+1}^{\infty} 2^{-n} \left(\|\varphi\|_{X^*} + \|\psi\|_{X^*}\right) \\ &\leq \sum_{n=N+1}^{\infty} 2^{-n+1} = 2^{-N+1} < \frac{\varepsilon}{2} \quad \text{for all } \psi \in B^*\end{aligned}$$

This implies in particular that

$$\left\{\psi \in B^* \,|\, \forall\, n \in \{1, 2, \ldots, N\} : |\varphi\left(x_n\right) - \psi\left(x_n\right)| < \frac{\varepsilon}{2}\right\} \subseteq O.$$

As $\varphi \in O$ was arbitrary, this ensures that $O \in \tau_{\mathrm{w}^*}$. As $O \in \tau_{\mathrm{d}}$ was arbitrary, we've arrived at showing $\tau_{\mathrm{d}} \subseteq \tau_{\mathrm{w}^*}$.

"$\tau_{\mathrm{w}^*} \subseteq \tau_{\mathrm{d}}$" Let $O \in \tau_{\mathrm{w}^*}$ and $\varphi \in O$ be arbitrary. Then there exist $N \in \mathbb{N}, \varepsilon \in (0, \infty)$ and $y_1, y_2, \ldots, y_N \in X$ satisfying that

$$\{\psi \in B^* \,|\, \forall\, n \in \{1, 2, \ldots, N\} : |\psi\left(y_n\right) - \varphi\left(y_n\right)| < \varepsilon\} \subseteq O$$

W.l.o.g. we may assume that $\sup_{n \in \mathbb{N}} \|y_n\|_X \leq 1$ (otherwise, replace y_n by $\frac{y_n}{\|y_n\|_X}$ if $\|y_n\|_X > 1$). Since $(x_n)_{n \in \mathbb{N}} \subseteq B$ is dense in B, there exist $k_1, k_2, \ldots, k_N \in \mathbb{N}$ such that

$$\|y_n - x_{k_n}\|_X < \frac{\varepsilon}{4} \quad \text{for all } n \in \{1, 2, \ldots, N\}.$$

Thus, with $\mathcal{N} := \max_{1 \leq i \leq N} k_i \in \mathbb{N}$, we have

$$\begin{aligned}&\left\{\psi \in B^* \,|\, \forall\, n \in \{1, 2, \ldots, \mathcal{N}\} : |\psi\left(x_n\right) - \varphi\left(x_n\right)| < \frac{\varepsilon}{2}\right\} \\ &\subseteq \{\psi \in B^* \,|\, \forall\, n \in \{1, 2, \ldots, N\} : |\psi\left(y_n\right) - \varphi\left(y_n\right)| < \varepsilon\}\end{aligned}$$

since, if $\psi \in B^*$ satisfies $|\psi(x_n) - \varphi(x_n)| < \frac{\varepsilon}{2}$ for all $n \in \{1, 2, \ldots, \mathcal{N}\}$, then it holds in particular for all $n \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned} |\psi(y_n) - \varphi(y_n)| &\leq |\psi(y_n) - \psi(x_{k_n})| + |\psi(x_{k_n}) - \varphi(x_{k_n})| + |\varphi(x_{k_n}) - \varphi(y_n)| \\ &\leq \|\psi\|_{X^*} \|y_n - x_{k_n}\|_X + |\psi(x_{k_n}) - \varphi(x_{k_n})| + \|\varphi\|_{X^*} \|x_{k_n} - y_n\|_X \\ &\leq 2\|y_n - x_{k_n}\|_X + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

But now we are done since for all $\psi \in B^*$ with $d(\psi, \varphi) < 2^{-\mathcal{N}}\frac{\varepsilon}{2}$ it holds that

$$|\psi(x_n) - \varphi(x_n)| \leq 2^n d(\psi, \varphi) < \frac{\varepsilon}{2} \quad \text{for all } n \in \{1, 2, \ldots, \mathcal{N}\}$$

which implies (having (1) in mind) that

$$\begin{aligned} &\left\{\psi \in B^* \mid d(\psi, \varphi) < 2^{-\mathcal{N}}\frac{\varepsilon}{2}\right\} \\ &\subseteq \left\{\psi \in B^* \,|\, \forall\, n \in \{1, 2, \ldots, \mathcal{N}\} : |\psi(x_n) - \varphi(x_n)\,| < \frac{\varepsilon}{2}\right\} \\ &\subseteq \{\psi \in B^* \,|\, \forall\, n \in \{1, 2, \ldots, N\} : |\psi(y_n) - \varphi(y_n)\,| < \varepsilon\}\,. \end{aligned}$$

As $\phi \in O$ was arbitrary, we demonstrated that $O \in \tau_{\mathrm{d}}$. As $O \in \tau_{\mathrm{w}^*}$ was arbitrary, we showed $\tau_{\mathrm{w}^*} \subseteq \tau_{\mathrm{d}}$

(b) The proof proceeds exactly as in (a).

□

Exercise 8.2

(a) Let $(X, \|\cdot\|_X)$ be a normed space and let $\emptyset \neq Q \subset X$ be an open, convex subset containing the origin. Prove that there exists a subset $\Upsilon \subset X^*$ such that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\},$$

which means that Q is an intersection of open, affine half-spaces.

(b) *Definition.* Let $(X, \|\cdot\|_X)$ be a normed space. The convex hull of $A \subset X$ is defined as

$$\operatorname{conv}(A) := \bigcap_{B \supset A, B \text{ convex}} B$$

Recall the following representation theorem for convex hulls

$$\operatorname{conv}(A) = \left\{ \sum_{k=1}^{n} \lambda_k x_k \mid n \in \mathbb{N}, x_1, \ldots, x_n \in A, \lambda_1, \ldots, \lambda_n \geq 0, \sum_{k=1}^{n} \lambda_k = 1 \right\}.$$

Using the representation of the convex hull above, prove that if $(X, \|\cdot\|_X)$ is a normed space and $A, B \subset X$ are compact, convex subsets, then $\operatorname{conv}(A \cup B)$ is compact.

Solution.

(a) Given the normed space $(X, \|\cdot\|_X)$, the non-trivial, open, convex subset $Q \subset X$ and the Minkowski functional

$$\begin{aligned} p : X &\to \mathbb{R} \\ x &\mapsto \inf\left\{\lambda > 0 \mid \frac{1}{\lambda} x \in Q\right\}, \end{aligned}$$

we define the set

$$\Upsilon := \{f \in X^* \mid \forall x \in X : f(x) \leq p(x)\}$$

and claim that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\}.$$

"$\subseteq$" Let $x \in Q$. Since Q is open, we have $p(x) < 1$. For every $f \in \Upsilon$ we have $f(x) \leq p(x)$ by definition. This proves $f(x) < 1$ for every $f \in \Upsilon$.

"$\supseteq$" Suppose $x_0 \notin Q$. We hope to find some $f \in \Upsilon$ with $f(x_0) \geq 1$. Towards that end, we define the functional

$$\begin{aligned} \ell : \operatorname{span}(\{x_0\}) &\to \mathbb{R} \\ t x_0 &\mapsto t. \end{aligned}$$

Since Q is convex and contains the origin, we have $p(x_0) \geq 1$. In particular, we have

$$\begin{aligned} \forall t \geq 0: \quad & \ell(tx_0) = t \leq tp(x_0) = p(tx_0) \\ \forall t < 0: \quad & \ell(tx_0) = t < 0 \leq p(tx_0) \end{aligned}$$

The Hahn-Banach theorem implies that there exists a linear functional $f : X \to \mathbb{R}$ which agrees with ℓ on $\operatorname{span}(\{x_0\})$ and satisfies $f(x) \leq p(x)$ for every $x \in X$. Is f continuous? Since Q is open and contains the origin, there exists $r > 0$ such that $B_r(0) \subset Q$. Thus, $\frac{1}{\lambda}x \in Q$ with $\lambda = \frac{2}{r}\|x\|_X$ and the definition of p implies that

$$f(x) \leq p(x) \leq \frac{2}{r}\|x\|_X$$

which yields that f is continuous and therefore $f \in \Upsilon$. Since $f(x_0) = 1$, the claim follows.

(b) For completeness, we first prove the representation of the convex hull in the statement.

Lemma 0.1. *The following representation theorem for convex hulls holds*

$$\operatorname{conv}(A) = \left\{ \sum_{k=1}^{n} \lambda_k x_k \mid n \in \mathbb{N}, x_1, \dots, x_n \in A, \lambda_1, \dots, \lambda_n \geq 0, \sum_{k=1}^{n} \lambda_k = 1 \right\}.$$

Proof. Given the normed space $(X, \|\cdot\|_X)$ and the subset $A \subset X$, let

$$\mathcal{C} := \left\{ \sum_{k=1}^{n} \lambda_k x_k \mid n \in \mathbb{N}, x_1, \dots, x_n \in A, \lambda_1, \dots, \lambda_n \geq 0, \sum_{k=1}^{n} \lambda_k = 1 \right\}$$

We prove $\operatorname{conv}(A) = \mathcal{C}$ by showing the two inclusions.

"$\subseteq$" Since $A \subset \mathcal{C}$, the inclusion $\operatorname{conv}(A) \subseteq \mathcal{C}$ follows from the definition of convex hull, if we show that $\mathcal{C}$ is convex. In fact, given $0 < t < 1$ we have

$$t \sum_{k=1}^{n} \lambda_k x_k + (1-t) \sum_{k=1}^{m} \lambda'_k x'_k = \sum_{k=1}^{n+m} \mu_k y_k$$

with

$$0 \leq \mu_k := \begin{cases} t\lambda_k & \text{if } k \in \{1, \dots, n\}, \\ (1-t)\lambda'_{k-n} & \text{if } k \in \{n+1, \dots, n+m\} \end{cases}$$

$$A \ni y_k := \begin{cases} x_k & \text{if } k \in \{1, \dots, n\}, \\ x'_{k-n} & \text{if } k \in \{n+1, \dots, n+m\} \end{cases}$$

and $\mu_1 + \ldots + \mu_{n+m} = t\,(\lambda_1 + \ldots + \lambda_n) + (1-t)\,(\lambda'_1 + \ldots + \lambda'_m) = t + (1-t) = 1$.

"$\supseteq$" Let $x_1, \dots, x_n \in A$ and let $\lambda_1, \dots, \lambda_n \geq 0$ with $\lambda_1 + \ldots + \lambda_n = 1$. We can assume $\lambda_1 \neq 0$. Since $\operatorname{conv}(A)$ is convex and contains $x_1, x_2 \in A$, and since $\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} = 1$,

$$\operatorname{conv}(A) \ni \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2 = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} =: y_2$$

For the same reason,

$$\operatorname{conv}(A) \ni \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} y_2 + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} x_3 = \frac{\lambda_1 x_2 + \lambda_2 x_2 + \lambda_3 x_3}{\lambda_1 + \lambda_2 + \lambda_3} =: y_3.$$

Iterating this procedure, we obtain

$$\operatorname{conv}(A) \ni \frac{\lambda_1 + \ldots + \lambda_{k-1}}{\lambda_1 + \ldots + \lambda_k} y_{k-1} + \frac{\lambda_k}{\lambda_1 + \ldots + \lambda_k} x_k = \frac{\lambda_1 x_1 + \ldots + \lambda_k x_k}{\lambda_1 + \ldots + \lambda_k} =: y_k.$$

for every $k \in \{3, \dots, n\}$. Since $\lambda_1 + \ldots + \lambda_n = 1$, we have $y_n = \lambda_1 x_1 + \ldots + \lambda_n x_n$ which concludes the proof of $\operatorname{conv}(A) \supseteq \mathcal{C}$. $\square$

Now we are ready to prove (b). Given the normed space $(X, \|\cdot\|_X)$, the convex subsets $A, B \subset X$ and defining $\triangle := \{(s,t) \in \mathbb{R}^2 \mid s + t = 1, s, t \geq 0\}$, we claim that

$$\operatorname{conv}(A \cup B) = \mathcal{D} := \bigcup_{(s,t) \in \triangle} (sA + tB)$$

"$\subseteq$" By choosing $(s,t) = (1,0)$ we see $A \subset \mathcal{D}$. Analogously, $B \subset \mathcal{D}$, hence $A \cup B \subset \mathcal{D}$. If $x \in (\operatorname{conv}(A \cup B)) \backslash (A \cup B)$, then the representation theorem for convex hulls implies that x is of the form

$$x = \sum_{k=1}^{j} s_k a_k + \sum_{k=j+1}^{n} t_k b_k,$$

where $0 \leq j \leq n \in \mathbb{N}$, where $a_k \in A, s_k \geq 0$ for all $k = 1, \dots, j$ and $b_k \in B, t_k \geq 0$ for every $k = j+1, \dots, n$, and where $s_1 + \dots + s_j + t_{j+1} + \dots + t_n = 1$. Since $x \notin A \cup B$ by assumption, we have

$$s := \sum_{k=1}^{j} s_k > 0, \quad t := \sum_{k=j+1}^{n} t_k > 0,$$

with $s + t = 1$. Since A and B are both convex by assumption,

$$a := \frac{1}{s} \sum_{k=1}^{j} s_k a_k \in A, \quad b := \frac{1}{t} \sum_{k=j+1}^{n} t_k b_k \in B,$$

and we have shown $x = sa + tb \in \mathcal{D}$.

"$\supseteq$" Let $a \in A$ and $b \in B$. Then $a, b \in \operatorname{conv}(A \cup B)$. Since $\operatorname{conv}(A \cup B)$ is convex, we must have $sa + tb \in \operatorname{conv}(A \cup B)$ for every $(s,t) \in \triangle$. This proves $\operatorname{conv}(A \cup B) \supseteq \mathcal{D}$.

Under the assumption that the convex sets A and B are compact, we show now that

$$\mathcal{D} = \bigcup_{(s,t) \in \triangle} (sA + tB)$$

is compact. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}$. Then there exist $a_n \in A$ and $b_n \in B$ as well as $(s_n, t_n) \in \triangle$ such that $x_n = s_n a_n + t_n b_n$ for every $n \in \mathbb{N}$. We argue in 3 steps:

- Since $\triangle$ is compact in $\mathbb{R}^2$, a subsequence $((s_n, t_n))_{n \in \Lambda_1 \subset \mathbb{N}}$ converges in $\triangle$.
- Since A is compact in X, a subsequence $(a_n)_{n \in \Lambda_2 \subset \Lambda_1}$ converges in A.
- Since B is compact in X, a subsequence $(b_n)_{n \in \Lambda_3 \subset \Lambda_2}$ converges in B.

Therefore, the subsequence $(x_n)_{n \in \Lambda_3}$ converges in $\mathcal{D}$ which concludes the proof.

□

Exercise 8.3 Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Given a positive integer n, consider n pairwise distinct points $x_1, \dots, x_n$ in X and the functional

$$F : X \to \mathbb{R}, \quad F(x) = \sum_{i=1}^{n} \|x - x_i\|_X^2$$

(a) Prove that the functional F has a global minimum on X, namely the value $\inf_{x\in X} F(x)$ is a real number attained by F at some $\bar{x} \in X$.

(b) Let us now assume that $(X, \|\cdot\|_X)$ is a Hilbert space (thus $\|\cdot\|_X$ is induced by a scalar product $\langle\cdot,\cdot\rangle_X$). Prove that the minimum $\bar{x} \in X$ is unique, and that $\bar{x}$ belongs to the convex hull K of $\{x_1, \ldots, x_n\}$.

Solution.

(a) First note that the map F is coercive, because $F(x) \geq \|x - x_1\|_X^2 \to \infty$ as $\|x\|_X \to \infty$. Moreover F is weakly sequentially lower semicontinuous because the map $x \mapsto \|x\|_X$ is.

Hence, since X is reflexive, the direct method (cf. "Variationsprinzip", Satz 5.4.1) applies and we obtain $\bar{x} \in X$ satisfying

$$F(\bar{x}) = \inf_{x\in X} F(x).$$

(b) Suppose, $\bar{y} \in X \setminus \{\bar{x}\}$ is another minimizer of F and consider $\bar{z} = \frac{1}{2}(\bar{x}+\bar{y})$. Since we are assuming that X is a Hilbert space, the parallelogram identity holds and implies

$$\begin{aligned}\|\bar{z} - x_i\|_X^2 &= \left\|\frac{\bar{x}-x_i}{2} + \frac{\bar{y}-x_i}{2}\right\|_X^2 \\ &= 2\left\|\frac{\bar{x}-x_i}{2}\right\|_X^2 + 2\left\|\frac{\bar{y}-x_i}{2}\right\|_X^2 - \underbrace{\| \frac{\bar{x}-x_i}{2} - \frac{\bar{y}-x_i}{2} \|_X^2}_{\neq 0} \\ &< \frac{\|\bar{x}-x_i\|_X^2}{2} + \frac{\|\bar{y}-x_i\|_X^2}{2}.\end{aligned}$$

Hence, a contradiction follows from

$$F(\bar{z}) < \frac{F(\bar{x})}{2} + \frac{F(\bar{y})}{2} = \inf_{x\in X} F(x)$$

which proves that the minimizer is unique.

Moreover, if $\|\cdot\|_X$ is induced by the scalar product $\langle\cdot,\cdot\rangle_X$, then the minimizer $\bar{x} \in X$ of F has the property that

$$\forall y \in X: \quad 0 = \frac{d}{dt}\bigg|_{t=0} F(\bar{x}+ty) = 2\sum_{i=1}^{n} \langle y, \bar{x} - x_i\rangle_X = 2\left\langle y, \sum_{i=1}^{n} (\bar{x} - x_i)\right\rangle_X$$

Consequently,

$$\sum_{i=1}^{n} (\bar{x} - x_i) = 0 \quad \Rightarrow \quad n\bar{x} = \sum_{i=1}^{n} x_i \quad \Rightarrow \quad \bar{x} = \sum_{i=1}^{n} \frac{1}{n} x_i$$

which proves that $\bar{x}$ is in the convex hull of $\{x_1, \ldots, x_n\} \subset X$.

$\square$

Exercise 8.4 Let $m \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^m$ be a bounded measurable set with $|\Omega| > 0$. For $g \in L^2(\mathbb{R}^m, \mathbb{R})$, we define the map

$$\begin{aligned} V : L^2(\Omega, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \int_\Omega \int_\Omega g(x-y) f(x) f(y) dy dx \end{aligned}$$

and given $h \in L^2(\Omega, \mathbb{R})$, we define the map

$$\begin{aligned} E : L^2(\Omega, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \|f - h\|^2_{L^2(\Omega,\mathbb{R})} + V(f). \end{aligned}$$

(a) Prove that V is weakly sequentially continuous by proceeding as follows.

(i) Show that the linear operator $T : L^2(\Omega, \mathbb{R}) \to L^2(\Omega, \mathbb{R})$ mapping $f \mapsto Tf$ given by

$$(Tf)(x) = \int_\Omega g(x-y) f(y) dy$$

is well-defined.

(ii) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega, \mathbb{R})$ such that $f_k \xrightarrow{w} f$ in $L^2(\Omega, \mathbb{R})$ as $k \to \infty$. Prove that $\|Tf_k - Tf\|_{L^2(\Omega,\mathbb{R})} \to 0$ as $k \to \infty$, where T is as in (i).

(iii) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega, \mathbb{R})$ such that $f_k \xrightarrow{w} f$ in $L^2(\Omega, \mathbb{R})$ as $k \to \infty$. Show that $V(f_k) \to V(f)$ as $k \to \infty$, i. e. V is weakly sequentially continuous.

(b) Under the assumption $g \geq 0$ almost everywhere, prove that E restricted to

$$L^2_+(\Omega, \mathbb{R}) := \left\{ f \in L^2(\Omega, \mathbb{R}) \mid f(x) \geq 0 \text{ for almost every } x \in \Omega \right\}$$

attains a global minimum.

Solution.

(a) Given a bounded measurable $\Omega \subseteq \mathbb{R}^m$ and $g \in L^2(\mathbb{R}^m, \mathbb{R})$, the goal is weak sequential continuity of the map

$$\begin{aligned} V : L^2(\Omega, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \int_\Omega \int_\Omega g(x-y) f(x) f(y) dy dx. \end{aligned}$$

(i) Let $f \in L^2(\Omega, \mathbb{R})$. Note that $(Tf)(x)$ is well-defined for every $x \in \Omega$ by the Cauchy-Schwarz inequality. Since $\Omega \subseteq \mathbb{R}^m$, being a bounded set, has finite volume $|\Omega| < \infty$, we obtain in addition that $Tf \in L^2(\Omega, \mathbb{R})$:

$$\begin{aligned} \|Tf\|_{L^2(\Omega,\mathbb{R})}^2 &= \int_\Omega |(Tf)(x)|^2 dx = \int_\Omega \left| \int_\Omega g(x-y) f(y) dy \right|^2 dx \\ &\leq \int_\Omega \left(\int_\Omega |g(x-y) f(y)| dy \right)^2 dx \\ &\leq \int_\Omega \left(\int_\Omega |g(x-y)|^2 dy \right) \|f\|_{L^2(\Omega,\mathbb{R})}^2 dx \\ &\leq \int_\Omega \|g\|_{L^2(\mathbb{R}^m,\mathbb{R})}^2 \|f\|_{L^2(\Omega,\mathbb{R})}^2 dx \leq |\Omega| \|g\|_{L^2(\mathbb{R}^m,\mathbb{R})}^2 \|f\|_{L^2(\Omega,\mathbb{R})}^2 < \infty. \end{aligned}$$

(ii) Since the sequence $(f_k)_{k\in\mathbb{N}}$ is weakly convergent, it is bounded (by Banach-Steinhaus: $\exists C \in [0, \infty)$ such that $\|f_k\|_{L^2(\Omega,\mathbb{R})} \leq C$ for every $k \in \mathbb{N}$. For every fixed $x_0 \in \Omega$ and $k \in \mathbb{N}$, there holds

$$\begin{aligned} |(Tf_k)(x_0)| &\leq \int_\Omega |g(x_0 - y) f_k(y)| dy \leq \left(\int_\Omega |g(x_0 - y)|^2 dy \right)^{\frac{1}{2}} \left(\int_\Omega |f_k(y)|^2 dy \right)^{\frac{1}{2}} \\ &\leq \|g\|_{L^2(\mathbb{R}^m,\mathbb{R})} \|f_k\|_{L^2(\Omega,\mathbb{R})} \end{aligned}$$

In particular, the map $f_k \mapsto (Tf_k)(x_0)$ is a linear continuous functional $L^2(\Omega, \mathbb{R}) \to \mathbb{R}$. Therefore, weak convergence $f_k \xrightarrow{w} f$ implies $(Tf_k)(x_0) \to (Tf)(x_0)$ as $k \to \infty$. In other words, Tf_k converges pointwise to Tf. Moreover,

$$\sup_{k\in\mathbb{N}} |(Tf_k)(x_0)| \leq \sup_{k\in\mathbb{N}} \left(\|g\|_{L^2(\mathbb{R}^m,\mathbb{R})} \|f_k\|_{L^2(\Omega,\mathbb{R})} \right) \leq C \|g\|_{L^2(\mathbb{R}^m,\mathbb{R})}.$$

Since Ω is bounded, the constant $C\|g\|_{L^2(\mathbb{R}^m,\mathbb{R})}$ on the right right hand side belongs $L^2(\Omega, \mathbb{R})$. Hence, the claim follows by Lebesgue's dominated convergence theorem.

(iii) Let T be as in Claim 1. Since $f_k \xrightarrow{w} f$ and $\|Tf_k - Tf\|_{L^2(\Omega,\mathbb{R})} \to 0$ as $k \to \infty$ by claim 2 , we conclude

$$V(f_k) = \int_\Omega f_k(x) \int_\Omega g(x-y) f_k(y) dy dx = \langle f_k, Tf_k \rangle_{L^2(\Omega)} \xrightarrow{k\to\infty} \langle f, Tf \rangle = V(f),$$

using the continuity property of scalar products proven in Exercise 6.3-(b).

(b) In the case that $0 \le g \in L^2(\mathbb{R}^m, \mathbb{R})$ and $h \in L^2(\Omega, \mathbb{R})$ the claim is that the map

$$\begin{aligned} E : L^2(\Omega, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \|f - h\|^2_{L^2(\Omega,\mathbb{R})} + V(f) \end{aligned}$$

restricted to $L^2_+(\Omega, \mathbb{R})$ attains a global minimum. Since $L^2(\Omega, \mathbb{R})$ is reflexive (being a Hilbert space), we may invoke the direct method in the calculus of variations if we prove the following claims.

Claim 1. $L^2_+(\Omega, \mathbb{R})$ is non-empty and weakly sequentially closed.

Proof. Clearly, $L^2_+(\Omega, \mathbb{R}) \ni 0$ is non-empty. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2_+(\Omega, \mathbb{R})$ such that $f_k \xrightarrow{w} f$ for some $f \in L^2(\Omega, \mathbb{R})$ as $k \to \infty$. Suppose $f \notin L^2_+(\Omega, \mathbb{R})$. Then there exists $U \subseteq \Omega$ with positive measure such that $f|_U < 0$. In particular, we can test the weak convergence with the characteristic function χ_U to obtain the contradiction

$$0 > \langle f, \chi_U \rangle_{L^2(\Omega,\mathbb{R})} = \lim_{k\to\infty} \langle f_k, \chi_U \rangle \ge 0$$

□

Claim 2. E: $L^2_+(\Omega, \mathbb{R}) \to \mathbb{R}$ is coercive and weakly sequentially lower semi-continuous.

Proof. Since $V(f) \ge 0$ if both $g \ge 0$ and $f \ge 0$ almost everywhere, we have

$$\begin{aligned} E(f) \ge \|f - h\|^2_{L^2(\Omega,\mathbb{R})} &\ge \|f\|^2_{L^2(\Omega,\mathbb{R})} - 2\|f\|_{L^2(\Omega,\mathbb{R})}\|h\|_{L^2(\Omega,\mathbb{R})} + \|h\|^2_{L^2(\Omega,\mathbb{R})} \\ &\ge \frac{1}{2}\|f\|^2_{L^2(\Omega,\mathbb{R})} - \|h\|^2_{L^2(\Omega,\mathbb{R})} \end{aligned}$$

for every $f \in L^2_+(\Omega, \mathbb{R})$ as we have by Young's inequality that $2ab \le \frac{1}{2}a^2 + 2b^2$ for all $a, b \in \mathbb{R}$. Since $h \in L^2(\Omega, \mathbb{R})$ is fixed, E is coercive. By part (a), $L^2(\Omega, \mathbb{R}) \ni f \mapsto V(f) \in \mathbb{R}$ is weakly sequentially lower semi-continuous. Moreover, every term on the right hand side of

$$\|f - h\|^2_{L^2(\Omega,\mathbb{R})} = \|f\|^2_{L^2(\Omega,\mathbb{R})} - 2\langle f, h \rangle_{L^2(\Omega,\mathbb{R})} + \|h\|^2_{L^2(\Omega,\mathbb{R})}$$

is weakly sequentially lower semi-continuous in f since h is fixed. This proves the claim. □

□