Exercise 9.1 Let X be a normed vector space and $Y \subset X$ a closed subspace. Write $\pi\colon X \to X/Y$ for the projection map $\pi(x) := x + Y$. Show that $\pi^*\colon (X/Y)^* \to X^*$ is injective, $\pi^*(\lambda) \in Y^\perp$ for all $\lambda \in (X/Y)^*$, and π^* induces an isometric isomorphism $(X/Y)^* \to Y^\perp$.

Exercise 9.2 Let X, Y be Banach spaces over $\mathbb{K}$, and let $A \in L(X, Y)$. The goal of this exercise is to give a direct proof of the Theorem in class about the relationship between range and kernel of A and A^*.

(a) Show that $\ker A^* = (\operatorname{ran} A)^\perp$ and $\ker A = {}^\perp(\operatorname{ran} A^*)$.

(b) Suppose that A has closed range. Prove the following statements.

  (i) There exists C such that for all $y \in \operatorname{ran} A$, there exists $x \in X$ with $Ax = y$ and $\|x\| \leq C\|y\|$.

  (ii) Given $\lambda \in (\ker A)^\perp$, define $\mu\colon \operatorname{ran} A \to \mathbb{K}$ by $\mu(Ax) = \lambda(x)$. Prove that μ is well-defined. Then show using (i) that μ is continuous. Let $\tilde{\mu} \in Y^*$ be an extension of μ (using Hahn–Banach). Show that $A^*\tilde{\mu} = \lambda$.

  (iii) Conclude that $\operatorname{ran} A^*$ is closed, and $\operatorname{ran} A = {}^\perp(\ker A^*)$ as well as $\operatorname{ran} A^* = (\ker A)^\perp$.

Exercise 9.3 Let $(H, (\cdot,\cdot))$ be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), let T be a continuous linear operator on H with $\|T\|_{L(H)} \leq 1$, let $U := \ker(I - T)$ (where I is the identity operator on H), let P_U denote the orthogonal projection onto U and let $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for every $n \in \mathbb{N}$. Our goal is to show that

$$\limsup_{n\to *\infty} \|S_n x - P_U x\|_H = 0 \qquad \forall\, x \in H.$$

For this, we recommend to proceed along the following steps:

(a) For all $x \in H$, we have $Tx = x$ if and only if $T^*x = x$.

(b) $U^\perp = \overline{\operatorname{ran}(I - T)}$.

(c) $\lim\limits_{n\to+\infty} S_n x = x$ for all $x \in U$ and $\lim\limits_{n\to+\infty} S_n x = 0$ for all $x \in U^\perp$.

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Exercise 9.4 Prove the following basic facts about compact operators.

(a) Finite rank operators are compact.

(b) If Y is a Hilbert space, then every compact operator $A \in L(X, Y)$ is a limit of finite rank operators.

(c) Hilbert–Schmidt operators are compact.

(d) Suppose X is reflexive, and $A \in L(X, Y)$. Suppose A maps every weakly convergent sequence into a norm convergent sequence. Show that A is compact.