Exercise 9.1 Let X be a normed vector space and $Y \subset X$ a closed subspace. Write $\pi\colon X \to X/Y$ for the projection map $\pi(x) := x + Y$. Show that $\pi^*\colon (X/Y)^* \to X^*$ is injective, $\pi^*(\lambda) \in Y^\perp$ for all $\lambda \in (X/Y)^*$, and π^* induces an isometric isomorphism $(X/Y)^* \to Y^\perp$.

Solution. First of all, we show that π^* is injective. Assume that $\lambda \in (X/Y)^*$ is such that $\pi^*(\lambda) = 0$. This means that

$$(\pi^*(\lambda))(x) = \lambda(\pi(x)) = 0, \qquad \forall\, x \in X.$$

But since π is surjective, this implies that $\lambda = 0$ in $(X/Y)^*$. Since π^* is linear, this sufficies to show that π^* is injective.

In order o show that $\pi^*(\lambda) \in Y^\perp$ for all $\lambda \in (X/Y)^*$, we need to show that

$$\pi^*(\lambda)(y) = \lambda(\pi(y)) = 0, \qquad \forall\, x \in Y.$$

This is clear, because λ is linear and $\pi(y) = 0 + Y$ for every $y \in Y$.

Since $(X/Y)^*$ and X^* are Banach spaces (they are dual spaces) and we have already shown that π^* is injective, in order to show that π^* induces an isomorphism $(X/Y)^* \to Y^\perp$ we just need to show that π^* is surjective onto $Y^\perp$. Hence, pick any $\varphi \in Y^\perp$ and define

$$\lambda(x + Y) := \varphi(x), \qquad \forall\, x + Y \in (X/Y)^*.$$

Since $\varphi \in Y^\perp$, we have that λ is well-defined. Indeed, given $x_1, x_2 \in X$ such that $x_1 + Y = x_2 + Y$, by definition there exists $y \in Y$ such that $x_2 - x_1 = y$. Hence

$$\lambda(x_1 + Y) = \varphi(x_1) = \varphi(x_1 + y) = \varphi(x_2) = \lambda(x_2 + Y).$$

Moreover, λ is continuous since

$$|\lambda(x + Y)| = |\leq |\varphi(x')| \leq \|\varphi\|_{X^*}\|x'\|_X, \qquad \forall\, x' \in X \text{ s.t. } x + Y = x' + Y.$$

By taking the infimum over the $x \in X$ such that $x + Y = x' + Y$, the continuity of λ follows. Finally, notice that

$$(\pi^*(\lambda))(x) = \lambda(\pi(x)) = \lambda(x + Y) = \varphi(x), \qquad \forall x \in X.$$

The statement follows. □

Exercise 9.2 Let X, Y be Banach spaces over $\mathbb{K}$, and let $A \in L(X, Y)$. The goal of this exercise is to give a direct proof of the Theorem in class about the relationship between range and kernel of A and A^*.

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(a) Show that $\ker A^* = (\operatorname{ran} A)^\perp$ and $\ker A = {}^\perp(\operatorname{ran} A^*)$.

(b) Suppose that A has closed range. Prove the following statements.

   (i) There exists C such that for all $y \in \operatorname{ran} A$, there exists $x \in X$ with $Ax = y$ and $\|x\| \leq C\|y\|$.

   (ii) Given $\lambda \in (\ker A)^\perp$, define $\mu\colon \operatorname{ran} A \to \mathbb{K}$ by $\mu(Ax) = \lambda(x)$. Prove that μ is well-defined. Then show using (i) that μ is continuous. Let $\tilde{\mu} \in Y^*$ be an extension of μ (using Hahn–Banach). Show that $A^*\tilde{\mu} = \lambda$.

   (iii) Conclude that $\operatorname{ran} A^*$ is closed, and $\operatorname{ran} A = {}^\perp(\ker A^*)$ as well as $\operatorname{ran} A^* = (\ker A)^\perp$.

Solution.

(a) First, we show that $\ker A^* = (\operatorname{ran} A)^\perp$. Indeed, assume that $\lambda \in \ker A^* \subset Y^*$. This holds if and only if

$$0 = (A^*(\lambda))(x) = \lambda(A(x)), \qquad \forall\, x \in X.$$

Again, the previous equality occurs if and only if $\lambda(y) = 0$ for every $y \in \operatorname{ran} A$, which is exactly as saying that $\lambda \in (\operatorname{ran} A)^\perp$. Hence, we have shown that $\ker A^* = (\operatorname{ran} A)^\perp$.

Second, to see that $\ker A = {}^\perp(\operatorname{ran} A^*)$ we fix any $x \in \ker A$ and we notice that this holds if and only if $(A^*(\lambda))(x) = \lambda(A(x)) = 0$ for every $\lambda \in Y^*$. But this is exactly as saying that $\mu(x) = 0$ for every $\mu \in \operatorname{ran} A^*$, which simply means that $x \in {}^\perp(\operatorname{ran} A^*)$. The statement follows.

(b)

   (i) Since A is continuous, we have that $\ker A$ is closed. Since A has closed range, the operator $\tilde{A} : X/\ker A \to \operatorname{ran} A$ given by $\tilde{A}(x + \ker A) = A(x)$ for every $X \in X$ is a well-defined isomorphism between Banach spaces. In particular, $\tilde{A}$ has a continuous inverse $\tilde{A}^{-1}$. Hence, there exists $C > 0$ such that

$$\|\tilde{A}^{-1}y\|_{X/\ker A} \leq C\|y\|_Y, \qquad \forall\, y \in \operatorname{ran} A.$$

   Recall that

$$\|\tilde{A}^{-1}y\|_{X/\ker A} = \inf\{\|x\|_X \text{ with } A(x) = y\}.$$

   In particular, by definition of infimum for every $y \in \operatorname{ran} A$ there exists $x \in X$ with $A(x) = y$ satisfying

$$\|x\|_X \leq 2\|\tilde{A}^{-1}y\|_{X/\ker A} \leq 2C\|y\|_Y.$$

   The statement follows.

(ii) To show that μ is well-defined, we notice that for every $x_1, x_2 \in X$ such that $Ax_1 = Ax_2 \Leftrightarrow A(x_1 - x_2) = 0$ it holds that

$$\mu(Ax_1) = \lambda(x_1) = \lambda(x_1 - x_2) + \lambda(x_2) = \lambda(x_2) = \mu(Ax_2),$$

since $\lambda \in (\ker A)^\perp$ and $x_1 - x_2 \in \ker A$.

In order to prove the continuity of μ, we notice that by (i) there exists $C > 0$ such that for every $y \in \operatorname{ran} A$ we find $x \in X$ with $Ax = y$ and $\|x\| \leq C\|y\|$. In particular, for this particular choice of x we have

$$|\mu(y)| = |\mu(Ax)| = |\lambda(x)| \leq \|\lambda\|_{Y^*}\|x\| \leq C\|\lambda\|_{Y^*}\|y\|, \qquad \forall\, y \in \operatorname{ran} A.$$

The continuity of μ follows.

Now assume that $\tilde{\mu}$ is some Hahn-Banach extension of μ to Y. We have

$$(A^*\tilde{\mu})(x) = \tilde{\mu}(Ax) = \mu(Ax) = \lambda(x), \qquad \forall\, x \in X.$$

This shows that $A^*\tilde{\mu} = \lambda$.

(iii) By (ii), we immediately conclude that $(\ker A)^\perp \subset \operatorname{ran} A^*$. On the other hand, by (i) we have that

$$(\ker A)^\perp = ({}^\perp(\operatorname{ran} A^*))^\perp = \overline{\operatorname{ran} A^*} \supset \operatorname{ran} A^*.$$

Hence, we get that $(\ker A)^\perp = \operatorname{ran} A^*$. Since the $^\perp$ of a subset is closed, we finally get that $\operatorname{ran} A^*$ is closed.

Moreover, since the $\operatorname{ran} A$ is closed, by (i) we have

$$\operatorname{ran} A = \overline{\operatorname{ran} A} = {}^\perp((\operatorname{ran} A)^\perp) = {}^\perp(\ker A^*).$$

□

Exercise 9.3 Let $(H, (\,\cdot, \cdot\,))$ be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), let T be a continuous linear operator on H with $\|T\|_{L(H)} \leq 1$, let $U := \ker(I - T)$ (where I is the identity operator on H), let P_U denote the orthogonal projection onto U and let $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for every $n \in \mathbb{N}$. Our goal is to show that

$$\limsup_{n\to*\infty} \|S_n x - P_U x\|_H = 0 \qquad \forall\, x \in H.$$

For this, we recommend to proceed along the following steps:

(a) For all $x \in H$, we have $Tx = x$ if and only if $T^*x = x$.

(b) $U^\perp = \overline{\operatorname{ran}(I-T)}$.

(c) $\lim_{n\to+\infty} S_n x = x$ for all $x \in U$ and $\lim_{n\to+\infty} S_n x = 0$ for all $x \in U^\perp$.

Solution.

(a) "⇒" Since $\|T^*\|_{L(H,H)} = \|T\|_{L(H,H)} \leq 1$, we have for all $x \in U$ (i.e. $x \in H$ with $Tx = x$) that

$$\|x\|_H \|T^*x\|_H \geq \langle x, T^*x\rangle = \langle Tx, x\rangle = \|x\|_H^2 \geq \|x\|_H \|T^*x\|_H \,,$$

which implies that $\|T^*x\|_H = \|x\|_H$ for all $x \in U$ (as well as $\langle Tx, x\rangle = \langle x, T^*x\rangle = \|x\|_H^2$ for all $x \in U$). Hence, we have for all $x \in U$ that

$$\|T^*x - x\|_H^2 = \|T^*x\|_H^2 - 2\operatorname{Re}\langle x, T^*x\rangle + \|x\|_H^2 = \|x\|_H^2 - 2\|x\|_H^2 + \|x\|_H^2 = 0.$$

Thus, $\ker(I-T) \subseteq \ker(I-T^*)$

"⇐" As $T^* \in L(H,H)$ also satisfies $\|T^*\|_{L(H,H)} \leq 1$, the argument above shows for all $x \in \ker(I-T^*)$ that $T^{**}x = x$. Since $T^{**} = T$ for every bounded linear operator on a Hilbert space, we have that $\ker(I-T) \supseteq \ker(I-T^*)$.

(b) We know from (a) that $U = \ker(I-T) = \ker(I-T^*)$. Hence, it holds that

$$U^\perp = (\ker(I-T^*))^\perp = \left(\operatorname{ran}(I-T)^\perp\right)^\perp = \overline{\operatorname{ran}(I-T)}$$

(c) For every $x \in U$, we have $Tx = x$, hence $S_n x = x$ for all $n \in \mathbb{N}$ and therefore $\limsup_{n\to\infty} \|S_n x - x\|_H = 0$. For every $x \in \operatorname{ran}(I-T)$, there exists $y \in H$ such that $x = (I-T)y$. Hence, it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned}\limsup_{n\to\infty} \|S_n x\|_H &= \limsup_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(y - Ty) \right\|_H \\ &= \limsup_{n\to\infty} \left\| \frac{1}{n}(y - T^n y) \right\|_H \leq \limsup_{n\to\infty} \frac{2\|y\|_H}{n} = 0.\end{aligned}$$

For every $x \in \overline{\operatorname{ran}(I-T)}$, there is a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \operatorname{ran}(I-T)$ converging to x as $n \to \infty$ and since $S_n y_k \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$, we get that

$$
\begin{aligned}
\limsup_{n\to\infty} \|S_n x\|_H &\leq \limsup_{n\to\infty} \left[\|S_n x - S_n y_k\|_H + \|S_n y_k\|_H\right] \\
&= \limsup_{n\to\infty} \|S_n x - S_n y_k\|_H \leq \limsup_{n\to\infty} \left[\|S_n\|_{L(H,H)} \|x - y_k\|_H\right] \\
&\leq \|x - y_k\|_H \qquad \text{for all } k \in \mathbb{N}.
\end{aligned}
$$

Hence, $\limsup_{n\to\infty} \|S_n x\|_H = 0$ for every $x \in \overline{\operatorname{ran}(I - T)} = U^\perp$. To come full circle, note that every $x \in H$ can be written as $x = (x - P_U x) + P_U x$, where $x - P_U x \in U^\perp$ and $P_U x \in U$, and therefore, we obtain for every $x \in H$ that $S_n x \to P_U x$ as $n \to \infty$ because $S_n (x - P_U x) \to 0$ and $S_n P_U x \to P_U x$ as $n \to \infty$.

□

Exercise 9.4 Prove the following basic facts about compact operators.

(a) Finite rank operators are compact.

(b) If Y is a Hilbert space, then every compact operator $A \in L(X, Y)$ is a limit of finite rank operators.

(c) Hilbert–Schmidt operators are compact.

(d) Suppose X is reflexive, and $A \in L(X, Y)$. Suppose A maps every weakly convergent sequence into a norm convergent sequence. Show that A is compact.

Solution.

(a) Let $T \in L(X, Y)$ be a finite rank operator. By continuity of T, we have that $T(B_X(0,1))$ is a bounded subset of the finite dimensional vector space $\operatorname{ran} T$. In particular, $T(B_X(0,1))$ is relatively compact by the Heine-Borel theorem and the statement follows.

(b) Let $A \in L(X, Y)$ be compact with Y Hilbert. Since $T(B_X(0,1))$ is relatively compact in Y, it is totally bounded. This means that for every $n \in \mathbb{N}$ there exists a cover of $T(B_X(0,1))$ by a finite number of open balls of radius $1/n$ centered at the points $y_1^n, ..., y_{N(n)}^n$. Denote by P_n the orthogonal projection (this is where we use the fact that Y is Hilbert) on the finite-dimensional subspace $\operatorname{span}\{y_1^n, ..., y_{N(n)}^n\}$. Define $T_n := P_n T$ for every $n \in \mathbb{N}$. Clearly, T_n is a finite rank operator for every $n \in \mathbb{N}$. Moreover, by construction, for every $n \in \mathbb{N}$ and for every $x \in B_X(0,1)$ there exists $y_x \in \{y_1^n, ..., y_{N(n)}^n\}$ such that

$$
\|Tx - y_x\|_Y \leq \frac{1}{n}.
$$

Hence,

$$\begin{aligned}\|T_n x - Tx\|_Y &\le \|T_n x - y_x\|_Y + \|Tx - y_x\|_Y \\ &\le \|P_n(Tx - y_x)\|_Y + \|Tx - y_x\|_Y \le 2\|Tx - y_x\|_Y \le \frac{2}{n}.\end{aligned}$$

This implies that

$$\|T_n - T\|_{L(X,Y)} \le \frac{2}{n}, \qquad \forall\, n \in \mathbb{N}.$$

The statement follows.

(c) Recall the definition of Hilbert-Schmidt operator from Exercise 6.1. Let T be Hilbert-Schmidt on the separable Hilbert space H and let $\{e_i\}_{i\in\mathbb{N}}$ be a complete orthonormal basis for H. For every $n \in \mathbb{N}$, define the linear operator $T_n : H \to \operatorname{span}\{e_1, ..., e_n\} \subset H$ by

$$T_n x := \sum_{k=0}^{n} (Te_k, e_k)_H (x, e_k)_H e_k, \qquad \forall\, x \in H.$$

Clearly, each T_n is a finite rank operator. Moreover, we claim that $T_n \to T$ in $L(H, H)$ as $n \to +\infty$. Indeed, by Parseval's identity, we have

$$\begin{aligned}\|T_n x - Tx\|_H^2 &= \sum_{k=0}^{+\infty} |(T_n x - Tx, e_k)_H|^2 = \sum_{k=n+1}^{+\infty} |(Te_k, e_k)_H|^2 |(x, e_k)_H|^2 \\ &\le \left(\sum_{k=n+1}^{+\infty} \|Te_k\|_H^2\right) \|x\|_H^2\end{aligned}$$

for every $x \in H$ and for every $n \in \mathbb{N}$. This implies that

$$\|T_n - T\|_{L(H,H)} \le \sum_{k=n+1}^{+\infty} \|Te_k\|_H^2 \to 0 \qquad (n \to +\infty).$$

The statement follows.

(d) Pick any sequence $\{Tx_n\}_{n\in\mathbb{N}}$ in $T_X(B_X(0,1))$. Since $\{x_n\}_{n\in\mathbb{N}} \subset B_X(0,1)$ is bounded and X is reflexive, we have that $\{x_n\}_{n\in\mathbb{N}}$ admits a weakly converging subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. But since T maps weakly converging sequences in strongly converging sequences we have that $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ is strongly convergent in Y. The statement follows.

□