Exercise Sheet 10

1. Regular Values

Let M and N be manifolds of the same dimension with M compact and let $f\colon M \to N$ be a smooth map. Let $y \in N$ be a regular value of f. Prove the following statements.

a) The preimage $f^{-1}(y)$ has only finitely many elements.

b) The number of elements in the fiber over y is locally constant in N. That is, for every regular value $y \in N$ there exists a neighborhood V of y, such that all $y' \in V$ are regular values and $\#f^{-1}(y) = \#f^{-1}(y')$.

c) If the space of regular values is connected, then $\#f^{-1}(y)$ is constant for all regular values.

2. Fundamental Theorem of Algebra

For a non-constant polynomial P over $\mathbb{C}$ we consider the map $\widetilde{P}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by $\widetilde{P}([z:1]) \coloneqq [P(z):1]$ and $\widetilde{P}([1:0]) \coloneqq [1:0]$.

a) Prove that $\widetilde{P}$ is a smooth map.

b) Prove that the space of regular values of $\widetilde{P}$ is connected.

c) Deduce the Fundamental Theorem of Algebra: every complex non-constant polynomial P has a zero in $\mathbb{C}$.

Hint: It suffices to show that $\widetilde{P}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is surjective.

3. Mapping Degree

Let $M \subset \mathbb{R}^3$ be a compact, connected surface (without boundary) with exterior Gauss map $N\colon M \to S^2$. Prove that

$$\deg(N) = \frac{1}{2}\chi(M).$$

Hint: Use Exercise 3 of Sheet 7.