Exercise Sheet 2

1. Characterization of convex curves

Let $c \in C^2([0,L],\mathbb{R}^2)$ be a simply C^2-closed curve parametrized by arc-length. Show that the following two statements are equivalent:

(i) The curvature κ_{or} of c doesn't change sign, that is, $\kappa_{\mathrm{or}}(t) \geq 0$ for all $t \in [0,L]$ or $\kappa_{\mathrm{or}} \leq 0$ for all $t \in [0,L]$.

(ii) The curve c is *convex*, that is, the image of c is the boundary of a convex subset $C \subset \mathbb{R}^2$.

2. Submanifolds

Prove that the following matrix groups are submanifolds of $\mathbb{R}^{n\times n}$:

(i) $\mathrm{SL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \; : \; \det A = 1\}$,

(ii) $\mathrm{SO}(n,\mathbb{R}) := \{A \in \mathrm{GL}(n,\mathbb{R}) \; : \; A^{-1} = A^{\mathrm{T}}, \det A = 1\}$.

3. Tangent bundle

Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. Show that the *tangent bundle*

$$TM := \bigcup_{p\in M} \{p\} \times TM_p$$

is a $2m$-dimensional submanifold of $\mathbb{R}^{2n}$.