D-MATH Differential Geometry I HS22
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Exercise Sheet 3

1. Differentiability

Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold, $F\colon M \longrightarrow \mathbb{R}^k$ a map and $p \in M$. Show that the following statements are equivalent.

(i) F is differentiable in p (as defined in the lecture).

(ii) There exist an open neighborhood V of p in $\mathbb{R}^n$ and a map $\bar{F}\colon V \longrightarrow \mathbb{R}^k$ differentiable in p with

$$\bar{F}|_{V\cap M} = F|_{V\cap M}.$$

2. Orientability

(i) Let $W \subset \mathbb{R}^n$ be an open set, $f\colon W \longrightarrow \mathbb{R}$ a C^1-map and $y \in \mathbb{R}$ a regular value of f. Prove that $M := f^{-1}(\{y\})$ is an orientable submanifold.

(ii) Prove that the submanifolds $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{R})$ are orientable.

3. Angle-preserving parametrization

Let $U \subset \mathbb{R}^m$ be an open set and $f\colon U \longrightarrow \mathbb{R}^n$ an immersion. The map f is called *angle-preserving* if for all $x \in U$ and $\xi, \eta \in TU_x$ the angles between ξ, η and $df_x(\xi), df_x(\eta)$ coincide. Here the angles are meant with respect to the standard scalar products on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

(i) Show that f is angle-preserving if and only if $g_{ij} = \lambda^2\delta_{ij}$, where g_{ij} is the matrix of the first fundamental form of f, δ_{ij} is the Kronecker delta and $\lambda\colon U \longrightarrow \mathbb{R}$ is a differentiable function.

(ii) Find an angle-preserving parametrization of the 2-sphere without North and South Pole, $S^2 \setminus \{N,S\} \subset \mathbb{R}^3$, of the form

$$f(x,y) = \Big(\sqrt{1-h^2(y)}\cos(x), \sqrt{1-h^2(y)}\sin(x), h(y)\Big),$$

where $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ is an odd function.