Exercise Sheet 4

1. Tubular surface

Let $c\colon\ [0,L] \to \mathbb{R}^3$ be a smooth Frenet curve, parametrized by arc-length, with normal vector n and binormal vector b. Show that if $r > 0$ is sufficiently small, then the tubular surface $f\colon\ [0,L] \times \mathbb{R} \to \mathbb{R}^3$ around c defined by

$$f(t,\varphi) \coloneqq c(t) + r\big(\cos\varphi \cdot n(t) + \sin\varphi \cdot b(t)\big)$$

is regular and the area of $f|_{[0,L]\times[0,2\pi)}$ equals $2\pi r L$.

2. Torus

Let $a > r > 0$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrization of a torus T, defined as

$$f(x,y) \coloneqq \big((a + r\cos x)\cos y, (a + r\cos x)\sin y, r\sin x\big).$$

Prove that:

(a) If a geodesic is at some point tangential to the circle $x = \frac{\pi}{2}$ then it must be contained in the region of T with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(b) Suppose that a geodesic $c\colon \mathbb{R} \longrightarrow T$, which crosses the circle $x = 0$ with an angle $\theta \in (0, \frac{\pi}{2})$, also intersects the circle $x = \pi$, then

$$\cos\theta \le \frac{a-r}{a+r}.$$

3. Energy

Let $M \subset \mathbb{R}^n$ be a submanifold, $c_0\colon\ [a,b] \to M$ a smooth curve and

$$E(c_0) \coloneqq \frac{1}{2}\int_a^b |\dot c_0(t)|^2 \,\mathrm{d}t$$

its energy.

(a) Show that $L(c_0)^2 \le 2(b-a)E(c_0)$ with an equality if and only if c_0 is parametrized proportionally to arc-length.

(b) Compute $\frac{d}{ds}\big|_{s=0} E(c_s)$ for a smooth variation $\{c_s\}_{s\in(-\varepsilon,\varepsilon)}$ of c_0 in M.

(c) Characterize geodesics in M using the energy.