Exercise Sheet 6

1. Parallel Surfaces

Given an immersion $f\colon U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$, with Gauss map $\nu\colon U \to S^2 \subset \mathbb{R}^3$ and $\varepsilon > 0$ we define $f^\varepsilon\colon U \to \mathbb{R}^3$ as

$$f^\varepsilon(x, y) \coloneqq f(x, y) + \varepsilon \cdot \nu(x, y).$$

Assuming that f has constant mean curvature $H \neq 0$ and non-vanishing Gauss curvature $K \neq 0$, show that the Gauss curvature of f^ε is constant for $\varepsilon = \frac{1}{2H}$.

2. Asymptotic Curves

Let $M \subset \mathbb{R}^3$ be a surface with $K < 0$. A curve $c\colon I \to M$ is called an *asymptotic curve* of M if $h_{c(t)}(\dot{c}(t), \dot{c}(t)) = 0$ for all $t \in I$. Prove that:

a) One can find a local parametrization of M whose parameter lines are asymptotic curves ("parametrization by asymptotic curves").

b) M is a minimal surface if and only if the asymptotic curves of a) are orthogonal to each other in every point.

3. Conjugate Minimal Surfaces

Let $U \subset \mathbb{R}^2$ be an open set. Two isothermally parametrized minimal surfaces $f, \tilde{f}\colon U \to \mathbb{R}^3$ are called *conjugate* if $f_1 = \tilde{f}_2$ and $f_2 = -\tilde{f}_1$.

a) Find isothermal parametrizations of the helicoid and the catenoid and show that they are conjugate.

b) Show that if f and $\tilde{f}$ are conjugate then $\{f^t\colon U \to \mathbb{R}^3\}_{t\in\mathbb{R}}$ with

$$f^t(x) \coloneqq \cos t \cdot f(x) + \sin t \cdot \tilde{f}(x)$$

is a family of isothermally parametrized minimal surfaces.

c) Show that the surfaces f^t are locally isometric to each other and find a Gauss map for f^t.