Exercise Sheet 8

1. Chebyshev Nets

Let $f\colon U \to \mathbb{R}^3$ be a parametrized surface with $U = (0, A) \times (0, B) \subset \mathbb{R}^2$.

a) Show that the following two conditions are equivalent:

   (i) For every rectangle $R = [u_1, u_1 + a] \times [u_2, u_2 + b] \subset U$ the opposite sides of $f(R)$ have the same length.

   (ii) $\frac{\partial g_{11}}{\partial u_2} = \frac{\partial g_{22}}{\partial u_1} \equiv 0$ on U.

   If f satisfies one of the equivalent conditions, then its parameter lines constitute a *Chebyshev net.*

b) Show that for such a parametrization there exists a change of coordinates $\varphi\colon U \to \tilde{U}$ such that the first fundamental form of $\tilde{f} := f \circ \varphi^{-1}$ has the form

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix},$$

   where ω is the angle between the parameter lines of f.

2. Isoperimetric problem on a Cartan-Hadamard surface

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface, such that f is a homeomorphism between $\mathbb{R}^2$ and $M := f(\mathbb{R}^2)$. Assume that f has nonnegative Gauss curvature K. Given $\Omega \subset M$ bounded, we say that $\partial\Omega$ is C^2 if it consists of a finite disjoint union of C^2 simple closed curves. For such Ω define the *isoperimetric quotient*

$$\mathcal{I}(\Omega) := \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)^{\frac{1}{2}}}$$

a) Suppose first that M is isometric to the Euclidean plane. Show that if Ω_0 is a minimizer of $\mathcal{I}$ (such that $\partial\Omega_0$ is C^2) then

$$\mathcal{I}(\Omega_0) = \sqrt{4\pi} \quad \text{and } \Omega_0 \text{ is an Euclidean disc.}$$

   Hint: Show that, by minimality, $\partial\Omega_0$ must consist of only one closed simple curve γ, and prove (using the first variation of arc length) that the geodesic curvature κ_g of γ must be constant. Deduce that γ must trace a circle in $\mathbb{R}^2$.

b) For general $K \leq 0$, show that if Ω_0 is a minimizer of $\mathcal{I}$ (with $\partial\Omega_0$ of class C^2) then it must be $K \equiv 0$ in Ω_0.

Hint: Using $\Omega_r = f(B_r(0))$, with $r \to 0^+$ as competitors, show that $\mathcal{I}(\Omega_0) \leq \sqrt{4\pi}$. Show that, as in a), $\partial\Omega_0$ must consist of only one closed simple curve γ. Let ν be the inwards unit normal to $\partial\Omega_0$, define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} I(\Omega_\varepsilon) \leq 0$, and < 0 unless $K \equiv 0$ in Ω_0.

3. The Brouwer Fixed Point Theorem

The Brouwer fixed point theorem states:

Theorem. *Let $D := \{x \in \mathbb{R}^n \ : \ |x| \leq 1\}$ be the unit ball. Then every continuous map $f \colon D \to D$ has a fixed point.*

a) Let $M \subset \mathbb{R}^3$ be a surface and $\tilde{D} \subset M$ a region diffeomorphic to the disc $D := \{x \in \mathbb{R}^2 \ : \ |x| \leq 1\}$. Consider a tangent vector field $X \colon \tilde{D} \to \mathbb{R}^3$ which on $\partial\tilde{D}$ is pointing outward. Show that X has zeros in the interior of $\tilde{D}$.

b) Prove the Brouwer fixed point theorem in two dimensions using part a).