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Exercise Sheet 9

1. Embeddings

a) Find an embedding $S^k \times \mathbb{R}^l \hookrightarrow \mathbb{R}^{k+l}$, where $k, l \geq 1$.

b) Prove that if the m-dimensional manifold M is a product of spheres, then there is an embedding $M \hookrightarrow \mathbb{R}^{m+1}$.

2. The Complex Projective Space

Consider the following equivalence relation on the complex vector space $\mathbb{C}^{n+1}$:

$$x \sim y \quad \Longleftrightarrow \quad x = \lambda y \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

The quotient space $\mathbb{CP}^n := \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\sim$, equipped with the quotient topology, is called *complex projective space.*

a) Find a differentiable structure on the topological space $\mathbb{CP}^n$ such that the canonical projection

$$\pi \colon \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$$

is a differentiable map.

b) Prove that S^2 and $\mathbb{CP}^1$ are diffeomorphic.

3. Hopf Fibration

Let $\pi \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the canonical projection from Exercise 2. The *Hopf fibration*

$$H \colon S^{2n+1} \to \mathbb{CP}^n$$

is given by the restriction of π to $S^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$

a) Let $n = 1$. Describe the fibers of H over a point $x \in \mathbb{CP}^1$, that is, $H^{-1}(x)$.

b) Prove that $H \colon S^{2n+1} \to \mathbb{CP}^n$ is a submersion.