Solutions 1

1. Arc length

Let $c \in C^1([0,1], \mathbb{R}^n)$. Show that the metric definition of arc length coincides with $L(c) := \int_0^1 |c'(t)| \, \mathrm{d}t$.

Solution. We'll denote by $l(c)$ the length of the curve c given by the metric definition.

We first show $l(c) \leq L(c)$. Let $0 = t_0 \leq \ldots \leq t_n = 1$ be a finite partition of $[0,1]$, then

$$\begin{aligned}\sum_{i=1}^{n} d(c(t_{i-1}), c(t_i)) = \sum_{i=1}^{n} |c(t_i) - c(t_{i-1})| &= \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} c'(\tau) \, \mathrm{d}\tau \right| \\ &\leq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |c'(\tau)| \, \mathrm{d}\tau = \int_0^1 |c'(\tau)| \, \mathrm{d}\tau,\end{aligned}$$

and thus $l(c) \leq L(c)$.

We now show the other inequality: let $\varepsilon > 0$ and choose $n \geq 2$ big enough such that $h := \frac{1}{n} < \varepsilon$. Consider the partition of $[0,1]$ given by $t_k := \frac{k}{n}$ for $k = 0, \ldots, n$, then

$$\begin{aligned}\frac{1}{h} \int_0^{1-h} d(c(t), c(t+h)) \, \mathrm{d}t &= \frac{1}{h} \int_0^{t_{n-1}} d(c(t), c(t+h)) \, \mathrm{d}t \\ &= \frac{1}{h} \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} d(c(t), c(t+h)) \, \mathrm{d}t \\ &= \frac{1}{h} \sum_{k=0}^{n-2} \int_0^h d(c(s+t_k), c(s+t_{k+1})) \, \mathrm{d}s \\ &= \frac{1}{h} \int_0^h \sum_{k=0}^{n-2} d(c(s+t_k), c(s+t_{k+1})) \, \mathrm{d}s \\ &\leq \frac{1}{h} \int_0^h l(c) \, \mathrm{d}s = l(c),\end{aligned}$$

where in the third equality we have used the substitution $s = t - t_k$. Using Fatou's lemma we obtain

$$\begin{aligned}\int_0^{1-\epsilon} |c'(t)| \, \mathrm{d}t &= \int_0^{1-\epsilon} \lim_{n\to\infty} \left| \frac{c(t+h) - c(t)}{h} \right| \mathrm{d}t \\ &\leq \liminf_{n\to\infty} \frac{1}{h} \int_0^{1-\epsilon} d(c(t), c(t+h)) \, \mathrm{d}t \leq l(c)\end{aligned}$$

and the statement follows by letting $\varepsilon \to 0$.

2. Osculating circle

Let $c \in C^2(I, \mathbb{R}^2)$ be a curve parametrized by arc length. A circle $S \subset \mathbb{R}^2$ with center $q \in \mathbb{R}^2$ and radius $r \geq 0$ is called *osculating circle* to c at the point $t \in I$ if S coincides with c at the point $c(t)$ up to second order.

Show that if $\ddot{c}(t) \neq 0$ then there is a unique osculating circle S to c at the point t. Find q, r and a parametrization α of S with $\alpha(t) = c(t), \dot{\alpha}(t) = \dot{c}(t)$ and $\ddot{\alpha}(t) = \ddot{c}(t)$.

Solution. We start with two remarks:

- Two curves α, β coincide up to second order at t_0 if
$$\alpha(t_0) = \beta(t_0), \qquad \dot{\alpha}(t_0) = \dot{\beta}(t_0), \qquad \ddot{\alpha}(t_0) = \ddot{\beta}(t_0).$$
- Every regular C^2-curve $c\colon I \to \mathbb{R}^2$ is a Frenet curve. If c is parametrized by arc-length then
$$\begin{aligned} e_1(t) &:= \dot{c}(t) \\ e_2(t) &:= e_1(t) \text{ rotated } \tfrac{\pi}{2} \text{ to the left.} \end{aligned}$$
From $\langle \dot{c}(t), \ddot{c}(t) \rangle = \frac{1}{2} \langle \dot{c}(t), \dot{c}(t) \rangle' = 0$ it follows that $\ddot{c}(t)$ and $e_2(t)$ are parallel and $\ddot{c}(t) = \kappa_{\mathrm{or}}(t) \cdot e_2(t)$. Therefore (for a Frenet curve)
$$\ddot{c}(t) \neq 0 \iff \kappa_{\mathrm{or}}(t) \neq 0.$$

We claim that the circle S with center
$$q \coloneqq c(t_0) + \frac{1}{\kappa_{\mathrm{or}}(t_0)} e_2(t_0)$$
and radius
$$r \coloneqq \frac{1}{|\kappa_{\mathrm{or}}(t_0)|}$$
is the unique osculating circle for c at t_0.

We parametrize S as follows
$$\alpha(t) = q + \frac{1}{\kappa_{\mathrm{or}}(t_0)} \Big(\sin\big(\kappa_{\mathrm{or}}(t_0)(t - t_0)\big) \cdot e_1(t_0) - \cos\big(\kappa_{\mathrm{or}}(t_0)(t - t_0)\big) \cdot e_2(t_0) \Big).$$
Then
$$\begin{aligned} \alpha'(t) &= \cos\big(\kappa_{\mathrm{or}}(t_0)(t - t_0)\big) \cdot e_1(t_0) + \sin\big(\kappa_{\mathrm{or}}(t_0)(t - t_0)\big) \cdot e_2(t_0), \\ \alpha''(t) &= \kappa_{\mathrm{or}}(t_0) \Big(- \sin\big(\kappa_{\mathrm{or}}(t_0)(t - t_0)\big) \cdot e_1(t_0) + \cos\big(\kappa_{\mathrm{or}}(t_0)(t - t_0)\big) \cdot e_2(t_0) \Big). \end{aligned}$$
At $t = t_0$ we have
$$\begin{aligned} \alpha(t_0) &= q - \frac{1}{\kappa_{\mathrm{or}}(t_0)} \cdot e_2(t_0) = c(t_0), \\ \dot{\alpha}(t_0) &= e_1(t_0) = \dot{c}(t_0), \\ \ddot{\alpha}(t_0) &= \kappa_{\mathrm{or}}(t_0) \cdot e_2(t_0) = \ddot{c}(t_0), \end{aligned}$$
and so S is an osculating circle for c at t_0.

We now prove uniqueness. Let T be another osculating circle for c at t_0 and denote by β an arc-length parametrization of T (with $\beta(t_0) = c(t_0)$, $\dot{\beta}(t_0) = \dot{c}(t_0)$ and $\ddot{\beta}(t_0) = \ddot{c}(t_0)$). Let a_1, a_2 be a Frenet frame for α and b_1, b_2 a Frenet frame for β, then
$$\beta(t_0) = c(t_0) = \alpha(t_0)$$

and

$$b_1(t_0) = \dot{\beta}(t_0) = \dot{c}(t_0) = \dot{\alpha}(t_0) = a_1(t_0),$$

so also $b_2(t_0) = a_2(t_0)$.

Moreover

$$\kappa_{\text{or},\beta}(t_0) \cdot b_2(t_0) = \ddot{\beta}(t_0) = \ddot{c}(t_0) = \ddot{\alpha}(t_0) = \kappa_{\text{or},\alpha}(t_0) \cdot a_2(t_0),$$

and hence $\kappa_{\text{or},\beta}(t_0) = \kappa_{\text{or},\alpha}(t_0)$.

Notice that circles have constant curvature κ, that is $\kappa(t_0) = \kappa(t)$, hence $\kappa_{\text{or},\alpha}(t) = \kappa_{\text{or},\beta}(t)$ for all t. It follows directly from the Fundamental Theorem of local curve theory that $\alpha(t) = \beta(t)$ and therefore $S = T$.

3. Curvature and torsion

a) Let $c \in C^3(I, \mathbb{R}^3)$ be a Frenet curve. Show that for the curvature κ and the torsion τ of c it holds that:

$$\kappa = \frac{|\dot{c} \times \ddot{c}|}{|\dot{c}|^3} \qquad \text{and} \qquad \tau = \frac{\det(\dot{c}, \ddot{c}, \dddot{c})}{|\dot{c} \times \ddot{c}|^2}.$$

b) Let $r, h > 0$ and denote by σ the following reflection of $\mathbb{R}^3$:

$$\sigma\colon \mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (x, y, -z).$$

Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the following Helixes:

$$\begin{aligned} c_1(t) &= (r\cos t, r\sin t, \tfrac{h}{2\pi}t), \\ c_2(t) &= c_1(-t), \\ c_3(t) &= \sigma \circ c_1(t). \end{aligned}$$

Solution.

a) From $e_1 = \frac{\dot{c}}{|\dot{c}|}$ it follows that $\dot{c} = |\dot{c}| \cdot e_1$ and from the first Frenet equation we have $\dot{e}_1 = |\dot{c}|\kappa \cdot e_2$, so

$$\ddot{c} = (|\dot{c}| \cdot e_1) = |\dot{c}|' \cdot e_1 + |\dot{c}| \cdot \dot{e}_1 = |\dot{c}|' \cdot e_1 + |\dot{c}|^2\kappa \cdot e_2$$

and

$$\begin{aligned} \dot{c} \times \ddot{c} &= |\dot{c}|' \cdot \dot{c} \times e_1 + |\dot{c}|2\kappa \cdot \dot{c} \times e_2 \\ &= (|\dot{c}|')^2 \cdot e_1 \times e_1 + |\dot{c}|^3\kappa \cdot e_1 \times e_2 \\ &= |\dot{c}|^3\kappa \cdot e_1 \times e_2 \\ &= |\dot{c}|^3\kappa \cdot e_3, \end{aligned}$$

thus $|\dot{c} \times \ddot{c}| = |\dot{c}|^3$, which solved for κ gives

$$\kappa = \frac{|\dot{c} \times \ddot{c}|}{|\dot{c}|^3}.$$

Moreover, using the above identity for $\dot e_1$ and the Frenet equation for $\dot e_2$ we obtain

$$\begin{aligned}\dddot{c} &= |\dot c|'' \cdot e_1 + |\dot c|' \cdot \dot e_1 + (|\dot c|^2\kappa)' \cdot e_2 + |\dot c|^2\kappa \cdot \dot e_2 \\ &= |\dot c|'' \cdot e_1 + \big(|\dot c|'|\dot c|\kappa + (|\dot c|^2\kappa)'\big) \cdot e_2 + |\dot c|^2\kappa \cdot \dot e_2 \\ &= |\dot c|'' \cdot e_1 + \big(|\dot c|'|\dot c|\kappa + (|\dot c|^2\kappa)'\big) \cdot e_2 + |\dot c|^2\kappa\big(- |\dot c|\kappa \cdot e_1 + |\dot c|\tau \cdot e_3\big) \\ &= \underbrace{\big(|\dot c|'' - |\dot c|^3\kappa^2\big)}_{=:A} \cdot e_1 + \underbrace{\big(|\dot c|'|\dot c|\kappa + (|\dot c|^2\kappa)'\big)}_{=:B} \cdot e_2 + \underbrace{|\dot c|^3\kappa\tau}_{=:C} \cdot e_3.\end{aligned}$$

Consequently we can compute $\det(\dot c, \ddot c, \dddot c)$ as follows:

$$\begin{aligned}\det(\dot c, \ddot c, \dddot c) &= \det\big(|\dot c| \cdot e_1, |\dot c|' \cdot e_1 + |\dot c|^2\kappa \cdot e_2, A \cdot e_1 + B \cdot e_2 + C \cdot e_3\big) \\ &= \det\big(|\dot c| \cdot e_1, |\dot c|^2\kappa \cdot e_2, |\dot c|^3\kappa\tau \cdot e_3\big) \\ &= |\dot c|^6\kappa^2\tau \det(e_1, e_2, e_3) \\ &= |\dot c|^6\kappa^2\tau \\ &= \tau \cdot |\dot c \times \ddot c|,\end{aligned}$$

which proves the statement.

b) We'll denote by $\kappa_1, \kappa_2, \kappa_2$ and τ_1, τ_2, τ_2 curvature and torsion of the curves c_1, c_2 and c_3, respectively.[1]

We compute

$$\begin{aligned} c_1(t) &= (r\cos t, r\sin t, \tfrac{h}{2\pi}t), \\ \dot c_1(t) &= (-r\sin t, r\cos t, \tfrac{h}{2\pi}), \\ \ddot c_1(t) &= (-r\cos t, -r\sin t, 0), \\ \dddot c_1(t) &= (r\sin t, -r\cos t, 0).\end{aligned}$$

It holds that

$$\begin{aligned} \dot c_1 \times \ddot c_1 &= (r\tfrac{h}{2\pi}\sin t, -r\tfrac{h}{2\pi}\cos t, r^2), \\ |\dot c_1 \times \ddot c_1| &= (r^2\tfrac{h^2}{4\pi^2} + r^4)^{\frac12} = r(\tfrac{h^2}{4\pi^2} + r^2)^{\frac12}, \\ |\dot c_1| &= (r^2 + \tfrac{h^2}{4\pi^2})^{\frac12},\end{aligned}$$

and therefore

$$\kappa_1 = \frac{|\dot c_1 \times \ddot c_1|}{|\dot c_1|^3} = \frac{r}{r^2 + \frac{h^2}{4\pi^2}}.$$

[1] Be careful! In the lecture we denoted $\kappa_1, \kappa_2, \ldots$ the different Frenet curvatures of a single curve c. In $\mathbb{R}^3$ *curvature* and *torsion* are simply defined as the first and second Frenet curvatures:

$$\kappa := \frac{1}{|\dot c|}\langle \dot e_1, e_2\rangle \qquad \text{and} \qquad \tau := \frac{1}{|\dot c|}\langle \dot e_2, e_3\rangle.$$

With

$$\det() = \det \begin{pmatrix} -r\sin t & -r\cos t & r\sin t \\ r\cos t & -r\sin t & -r\cos t \\ \frac{h}{2\pi} & 0 & 0 \end{pmatrix}$$
$$= r^2 \frac{h}{2\pi}\cos^2 t + r^2 \frac{h}{2\pi}\sin^2 t = r^2 \frac{h}{2\pi}$$

it follows that

$$\tau_1 = \frac{\det\big(\dot c_1, \ddot c_1, \dddot c_1\big)}{|\dot c_1 \times \ddot c_1|^2} = \frac{r^2 \frac{h}{2\pi}}{r^2(\frac{h^2}{4\pi^2} + r^2)} = \frac{\frac{h}{2\pi}}{\frac{h^2}{4\pi^2} + r^2}.$$

For c_2 we have

$$\begin{aligned} c_2(t) &= c_1(-t) \\ \dot c_2(t) &= -\dot c_1(-t), \\ \ddot c_2(t) &= \ddot c_1(-t), \\ \dddot c_2(t) &= -\dddot c_1(-t), \end{aligned}$$

therefore

$$\kappa_2(t) = \frac{|\dot c_2(t) \times \ddot c_2(t)|}{|c_2'(t)|^3} = \frac{|-\dot c_1(-t) \times \ddot c_1(-t)|}{|-\dot c_1(-t)|^3} = \kappa_1(-t) = \frac{r}{r^2 + \frac{h^2}{4\pi^2}}$$

and

$$\begin{aligned} \tau_2(t) &= \frac{\det\big(\dot c_2(t), \ddot c_2(t), \dddot c_2(t)\big)}{|\dot c_2(t) \times \ddot c_2(t)|^2} \\ &= \frac{\det\big(-\dot c_1(-t), \ddot c_1(-t), -\dddot c_1(-t)\big)}{|-\dot c_1(-t) \times \ddot c_1(-t)|^2} = \tau_1(-t) = \frac{\frac{h}{2\pi}}{\frac{h^2}{4\pi^2} + r^2}. \end{aligned}$$

The curve c_3 satisfies $|\dot c_3 \times \ddot c_3| = |\dot c_1 \times \ddot c_1|$ and $\det(\dot c_3, \ddot c_3, \dddot c_3) = -\det(\dot c_1, \ddot c_1, \dddot c_1)$, so

$$\begin{aligned} \kappa_3 &= \kappa_2 = \frac{r}{r^2 + \frac{h^2}{4\pi^2}}, \\ \tau_3 &= -\tau_1 = -\frac{\frac{h}{2\pi}}{\frac{h^2}{4\pi^2} + r^2}. \end{aligned}$$