Solutions 10

1. Regular Values

Let M and N be manifolds of the same dimension with M compact and let $f\colon M \to N$ be a smooth map. Let $y \in N$ be a regular value of f. Prove the following statements.

a) The preimage $f^{-1}(y)$ has only finitely many elements.

b) The number of elements in the fiber over y is locally constant in N. That is, for every regular value $y \in N$ there exists a neighborhood V of y, such that all $y' \in V$ are regular values and $\#f^{-1}(y) = \#f^{-1}(y')$.

c) If the space of regular values is connected, then $\#f^{-1}(y)$ is constant for all regular values.

Solution. a) As y is a regular value of f, we know that df_x is surjective for all $x \in f^{-1}(y)$ and since M and N have the same dimension df_x is bijective. Therefore f is locally a diffeomorphism, that is, there exist an open neighborhood U_x of x and an open neighborhood V_x of y such that $f|_{U_x}\colon U_x \to V_x$ is a diffeomorphism. In particular $U_x \cap f^{-1}(y) = \{x\}$.

Moreover f is continuous and $\{y\}$ is closed in N, so $f^{-1}(y) \subset M$ is closed and hence compact. This implies that the open cover $\{U_x\}_{x\in f^{-1}(y)}$ of $f^{-1}(y)$ admits a finite subcover $\{U_{x_i}\}_{i=1}^n$. Together with the above observation we conclude that

$$f^{-1}(y) = f^{-1}(y) \cap \bigcup_{i=1}^n U_{x_i} = \bigcup_{i=1}^n (U_{x_i} \cap f^{-1}(y)) = \bigcup_{i=1}^n \{x_i\} = \{x_1, \dots, x_n\}$$

and hence $f^{-1}(y)$ is finite.

b) Let $f^{-1}(y) = \{x_1, \dots, x_n\}$ with $U_i \ni x_i$ open and $U_i \cap U_j = \emptyset$, such that $f|_{U_i}\colon U_i \to V_i$ is a diffeomorphism (as above, restricting the neighborhoods to make them pairwise disjoint, if necessary).

The set $A \coloneqq M \setminus \bigcup_{i=1}^n U_i$ is closed and hence compact. It follows that $f(A)$ is compact and thus closed. Hence its complement $W \coloneqq N \setminus f(A)$ is open and $y \in W$, since $A \cap f^{-1}(y) = \emptyset$.

We define $V \coloneqq \bigcap_{i=1}^n V_i \cap W$, which is an open neighborhood of y with the additional property that for all $y' \in V$ the preimage is contained in $\bigcup_{i=1}^n U_i$. Therefore it holds that

$$f^{-1}(y') = f^{-1}(y') \cap \bigcup_{i=1}^n U_i = \bigcup_{i=1}^n (U_i \cap f^{-1}(y')) = \bigcup_{i=1}^n \{x'_i\} = \{x'_1, \dots, x'_n\},$$

so $\#f^{-1}(y) = \#f^{-1}(y')$. Moreover all the x_i' (and hence the y') are regular, as $f|_{U_i}\colon U_i \to V_i$ is a diffeomorphism.

c) Let R be the set of all regular values of f. From b) it follows that the map $g\colon R \to \mathbb{Z}$, $y \mapsto \#f^{-1}(y)$ is continuous. Thus $g(R)$ is connected, that is, g is constant on R.

2. Fundamental Theorem of Algebra

For a non-constant polynomial P over $\mathbb{C}$ we consider the map $\widetilde{P}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by $\widetilde{P}([z:1]) \coloneqq [P(z):1]$ and $\widetilde{P}([1:0]) \coloneqq [1:0]$.

a) Prove that $\widetilde{P}$ is a smooth map.

b) Prove that the space of regular values of $\widetilde{P}$ is connected.

c) Deduce the Fundamental Theorem of Algebra: every complex non-constant polynomial P has a zero in $\mathbb{C}$.

Hint: It suffices to show that $\widetilde{P}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is surjective.

Solution. a) Suppose that $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ with $a_0 \neq 0$.

Notice that $\widetilde{P}(U_1) \subset U_1$ and $\widetilde{P}$ is smooth on $\mathbb{CP}^1 \setminus \{[1:0]\} = U_1$ since

$$\phi_1 \circ \widetilde{P} \circ \phi_1^{-1}(z) = P(z).$$

It remains to check the smoothness of $\widetilde{P}$ at $[1:0] \in U_0 = \phi_0^{-1}(\mathbb{C})$. Note that $\widetilde{P}([1:0]) = [1:0] \in U_0$ and $\phi_0 \circ \widetilde{P} \circ \phi_0^{-1}(0) = 0$.

Moreover for $z \neq 0$ $\phi_0^{-1}(z) = [1:z] = \left[\frac{1}{z}:1\right]$ and $P(\frac{1}{z}) = \frac{1}{z^n}(a_0 + \cdots + a_n z^n)$, which is not zero for z in a neighborhood of $0 \in \mathbb{C}$ small enough. Thus $\widetilde{P} \circ \phi_0^{-1}(z) = \widetilde{P}(\left[\frac{1}{z}:1\right]) = \left[P(\frac{1}{z}):1\right] \in U_0$ and

$$\phi_0 \circ \widetilde{P} \circ \phi_0^{-1}(z) = \frac{1}{P(\frac{1}{z})} = \frac{z^n}{a_0 + a_1 z + \ldots + a_n z^n}.$$

Both expressions for $z = 0$ and $z \neq 0$ (in a neighborhood of 0) coincide and show that $\phi_0 \circ \widetilde{P} \circ \phi_0^{-1}$ is smooth around 0, hence $\widetilde{P}$ is smooth around $[1:0]$.

b)Singular points of $\widetilde{P}$ in $\mathbb{CP}^1 \setminus \{[1:0]\}$ correspond to zeroes of $P'(z)$. As these are only finitely many, it follows that $\widetilde{P}$ as only finitely many singular values $\{y_1, \ldots, y_n\}$. Then it is known that $\mathbb{CP}^1 \setminus \{y_1, \ldots, y_n\} \cong S^2 \setminus \{y_1', \ldots, y_n'\}$ is connected.

c) From b) and Exercise 2b) we know that $\#\widetilde{P}^{-1}(y)$ is constant for all regular values. Suppose that $\widetilde{P}$ is not surjective. Then there exists $y' \in \mathbb{CP}^1$

with $\widetilde{P}^{-1}(y') = \emptyset$. This is a regular value and hence we obtain $\widetilde{P}^{-1}(y) = \emptyset$ for all regular values $y \in \mathbb{CP}^1$. But then $\widetilde{P}(\mathbb{CP}^1)$ is finite (see b) and so P is constant. Contradiction, thus $\widetilde{P}$ is surjective.

In particular there exists $z \in \mathbb{C}$ with $\widetilde{P}([z:1]) = [P(z):1] = [0:1]$, that is, $P(z) = 0$.

3. Mapping Degree

Let $M \subset \mathbb{R}^3$ be a compact, connected surface (without boundary) with exterior Gauss map $N\colon M \to S^2$. Prove that

$$\deg(N) = \frac{1}{2}\chi(M).$$

Hint: Use Exercise 3 of Sheet 7.

Solution. Note that $p \in M$ is a regular point of N if and only if $K(p) \neq 0$, since $K(p) = \det(-dN_p)$. Moreover

$$\operatorname{sgn}(dN_p) = \begin{cases} +1, & K(p) > 0, \\ -1, & K(p) < 0. \end{cases}$$

We define $M_+ \coloneqq \{p \in M : K(p) > 0\}$ and $M_- \coloneqq \{p \in M : K(p) < 0\}$.

By the Theorem of Gauss-Bonnet and Exercise 3 of Sheet 7 we obtain

$$2\pi\chi(M) = \int_M K \,\mathrm{d}A = \int_{M_+} |K| \,\mathrm{d}A - \int_{M_-} |K| \,\mathrm{d}A = A(N|_{M_+}) - A(N|_{M_-}).$$

Now, let $R \subset S^2$ be set of all regular values of N.

The area of N is counted with multiplicities and from Sard's Theorem almost every value of N is regular, hence we compute

$$\begin{aligned} A(N|_{M_+}) - A(N|_{M_-}) &= \int_{N(M_+)} \#N|_{M_+}^{-1}(q) dA(q) - \int_{N(M_-)} \#N|_{M_-}^{-1}(q) dA(q) \\ &= \int_{N(M_+)\cap R} \#N|_{M_+}^{-1}(q) dA(q) - \int_{N(M_-)\cap R} \#N|_{M_-}^{-1}(q) dA(q) \\ &= \int_R \underbrace{\Big(\#N|_{M_+}^{-1}(q) - \#N|_{M_-}^{-1}(q)\Big)}_{=\deg N} dA(q) \\ &= A(S^2) \deg N = 4\pi \deg N, \end{aligned}$$

hence $2\pi\chi(M) = 4\pi \deg N$ and therefore $\deg(N) = \frac{1}{2}\chi(M)$.