Solution 11

For each of the following multiple choice questions, choose **the** correct answer.

1. The radius of the osculating circle at a point of the helix $c(t) := (r\cos t, r\sin t, t)$ is:

(a) $\frac{r}{1+r^2}$.

√ (b) $\frac{1+r^2}{r}$.

(c) r.

(d) $\frac{1}{r}$.

(e) $\frac{\sqrt{1+r^2}}{r}$.

Solution. The parametrization has constant speed $|c'| = |(-r\sin t, r\cos t, 1)| = \sqrt{1+r^2}$. The accelaration for a unit speed parametrization is then $\frac{|c''|}{1+r^2} = \frac{r}{1+r^2}$ and the radius of curvature is the inverse of this.

2. Let M be a smooth surface in $\mathbb{R}^3$ with Gauss curvature K and mean curvature H. Which of the following relations is always true?

(a) $H \geq K$.

(b) $H \leq K$.

(c) $H^2 \leq K$.

√ (d) $H^2 \geq K$.

(e) $H^2 = K^2$.

Solution. If k_1, k_2 denote the principal curvatures of M, then $K = k_1 k_2$ and $H = \frac{1}{2}(k_1+k_2)$. We compute

$$4H^2 = (k_1+k_2)^2 = k_1^2 + k_2^2 + 2k_1k_2 = (k_1-k_2)^2 + 4k_1k_2 \geq 4k_1k_2 = 4K.$$

3. Consider the following curve in $\mathbb{R}^3$:

$$\gamma(t) = \big(3\cos(t/5), 4\cos(t/5), 5\sin(t/5)\big).$$

Which of the following vectors is the binormal B of γ?

√ (a) $(\frac{4}{5}, -\frac{3}{5}, 0)$.

(b) $(0, \frac{4}{5}, -\frac{3}{5})$.

(c) $(1, 0, 0)$.

(d) $(-\frac{3}{5}, \frac{4}{5})$.

(e) $\big(-\frac{3}{5}\sin(\frac{t}{5}), -\frac{4}{5}\sin(\frac{t}{5}), \cos(\frac{t}{5})\big)$.

Solution. We compute

$$\begin{aligned}
\gamma'(t) &= \big(-\tfrac{3}{5}\sin(\tfrac{t}{5}), -\tfrac{4}{5}\sin(\tfrac{t}{5}), \cos(\tfrac{t}{5})\big)\\
\gamma''(t) &= \big(-\tfrac{3}{25}\cos(\tfrac{t}{5}), -\tfrac{4}{25}\cos(\tfrac{t}{5}), -\tfrac{1}{5}\sin(\tfrac{t}{5})\big)\\
|\gamma''(t)| &= \big((\tfrac{9}{25^2} + \tfrac{16}{25^2})\cos(\tfrac{t}{5})^2 + \tfrac{1}{25}\sin(\tfrac{t}{5})^2\big)^{1/2} = \tfrac{1}{5}
\end{aligned}$$

Thus the Frenet frame $(T, N, B) = (e_1, e_2, e_3)$ of γ is given by

$$\begin{aligned}
e_1(t) &= \gamma'(t) = \big(-\tfrac{3}{5}\sin(\tfrac{t}{5}), -\tfrac{4}{5}\sin(\tfrac{t}{5}), \cos(\tfrac{t}{5})\big)\\
e_2(t) &= \frac{\gamma''(t)}{|\gamma''(t)|} = \big(-\tfrac{3}{5}\cos(\tfrac{t}{5}), -\tfrac{4}{5}\cos(\tfrac{t}{5}), -\sin(\tfrac{t}{5})\big)\\
e_3 &= e_1 \times e_2 = (\tfrac{4}{5}, -\tfrac{3}{5}, 0).
\end{aligned}$$

4. Let $C \subset \mathbb{R}^3$ be the cylinder in $\mathbb{R}^3$, parametrized as

$$f(u,v) = \big(\cos(u), \sin(u), v\big).$$

What are the correct values of the Gauss curvature K and mean curvature H at the point $(\sqrt{2}/2, \sqrt{2}/2, 100) \in C$ (with respect to the outward pointing Gauss map) ?

(a) $K = 0, H = 0$.

√ (b) $K = 0, H = -\frac{1}{2}$.

(c) $K = -1, H = 0$.

(d) $K = 0, H = 1$.

(e) $K = 1, H = -1$.

Solution The principal curvatures are $k_1 = -1$ and $k_2 = 0$, thus $K = 0$ and $H = -\frac{1}{2}$.

5. Consider a "quadrilateral" region of area A in a 2-sphere of radius r (connected region bounded by four great circular arcs). The sum of its interior angles is:

(a) $2\pi - A/r^2$.

(b) $\pi - A$.

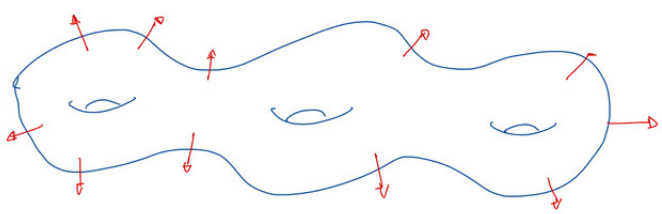
√ (c) $2\pi + A/r^2$.

(d) $2\pi r^2 + A$.

(e) $2\pi(1 + A/r^2)$.

Solution The sum of the angles of a spherical triangle in a sphere of radius r is $\pi + A/r^2$.

6. Let $M \subset \mathbb{R}^3$ be the following smooth surface:



What is the degree of the outward pointing normal red vector field X?

(a) $\deg(X) = 0$.

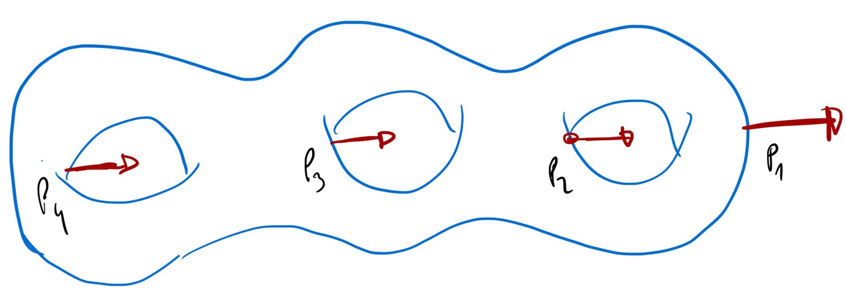
(b) $\deg(X) = 1$.

(c) $\deg(X) = 2$.

(d) $\deg(X) = -1$.

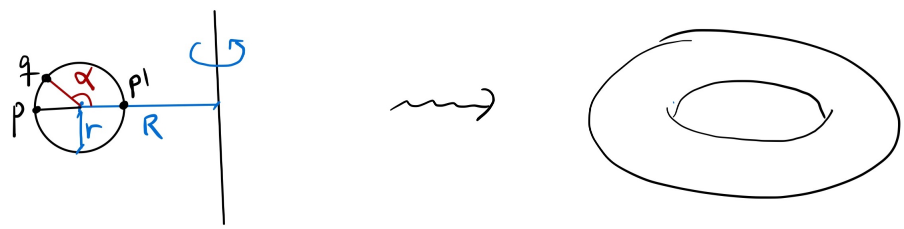
√ (e) $\deg(X) = -2$.

Solution Recall that the degree is invariant under smooth homotopy and independent of the point (of S^2 in this case) chosen to compute it. So, it is enough to compute the degree for our favourite drawing of the surface and choosing the value $\nu_0 \in S^2$ at our best convenience. For instance, looking at the figure



we see that $\{p_1, p_2, p_3, p_4\}$ is the pre-image of some ν_0 on the sphere. Now, the Gaussian curvature at p_1 is > 0, but the Gaussian curvature is < 0 at p_2, p_3, p_4. So the degree is $1 - 3 = -2$.

7. Consider the torus of revolution $f(x,y) = \big(\cos x(-R+r\cos y), \sin x(-R+r\cos y), r\sin y\big)$, $R > r$, drawn below:



Its Gauss' curvatures at $p = (-R-r, 0, 0)$ and $p' = (-R+r, 0, 0)$ are

(a) $\frac{1}{r\sqrt{R^2+r^2}}$ and $\frac{-1}{r\sqrt{R^2-r^2}}$, resp.

(b) $\frac{1}{rR}$ and $\frac{-1}{rR}$, resp.

(c) Both equal, in absolute value, to $\frac{1}{\sqrt{rR}}$.

(d) Both equal, in absolute value, to $\frac{1}{rR}$.

√ (e) $\frac{1}{r(R+r)}$ and $\frac{-1}{r(R-r)}$, resp.

Solution At the point p both principal curvatures have the same sign. One is equal in absolute value to $\frac{1}{r}$ and the other to $\frac{1}{R+r}$. At p' the principal curvatures have opposite signs. One is equal in absolute value to $\frac{1}{r}$ and the other to $\frac{1}{R-r}$.

8. Consider again the torus from question 7. The mean curvature at the point $q = (-R + r\cos\alpha, 0, r\sin\alpha)$ with respect to the outwards normal (pointing towards the unbounded component of $\mathbb{R}^3 \setminus f([0,2\pi]^2)$) is:

(a) $\frac{1}{2}\big(-\frac{1}{r} + \frac{1}{R - r\cos\alpha}\big)$.

(b) $\frac{1}{2}\big(-\frac{1}{r} + \frac{\sin\alpha}{R - r\cos\alpha}\big)$.

(c) $\frac{1}{2}\big(-\frac{1}{r} + \frac{\tan\alpha}{R - r}\big)$.

√ (d) $\frac{1}{2}\big(-\frac{1}{r} + \frac{\cos\alpha}{R - r\cos\alpha}\big)$.

(e) $\frac{1}{2}\big(-\frac{1}{r} + \frac{\tan\alpha}{R + r}\big)$.

Solution One principal curvature at any point is $-\frac{1}{r}$. To compute the orthogonal one at q we consider the circle $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$. Its curvature is $1/(R - r\cos\alpha)$, so the normal curvature is $\frac{\cos\alpha}{R - r\cos\alpha}$. The mean curvature is then the average of $-\frac{1}{r}$ and $\frac{\cos\alpha}{R - r\cos\alpha}$.

9. Consinder again the torus from question 7. When the point q is rotated about the x_3 axis it generates the curve $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$, which is contained in the torus. Given a tangent vector X at q consider its parallel transport along γ for one full turn ($t \in [0, 2\pi]$), producing a new tangent vector Y at q. The angle between X and Y is:

(a) $\frac{\alpha R}{r}$.

√ (b) $2\pi\sin\alpha$.

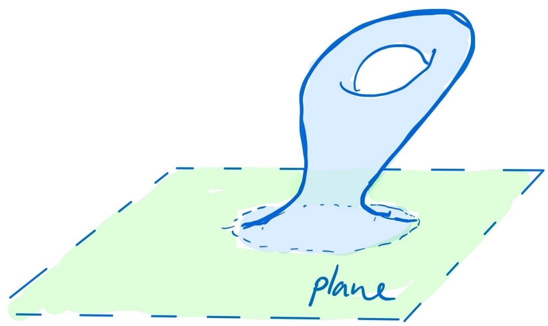
(c) $\frac{\tan\alpha R}{r}$.

(d) $2\pi\cos\alpha$.

(e) $\sin\alpha$.

Solution Consider the cone tangent to the torus along γ. It is a cone of revolution (also with respect the x_3 axis) and the angle of its generating lines of the cone and the x_3 axis is $\sin\alpha$. Hence when “opening” the cone (as we saw in the lecture in the it becomes a flat example of Foucault’s pendulum) it becomes a flat circular sector of angle $2\pi\sin\alpha$. Hence, since parallel transport is trivial for the flat surface, we see that the angle between the transported vector and the original one is $2\pi\sin\alpha$.

10. Consider a smooth surface S obtained by gluing a torus (minus a disk) and a rectangular piece of plane (minus a disk), as in the figure. While the torus part was stretched in order to be tangent to the plane, the planar part was kept exactly flat.



Then $\int_S K dA$ is

(a) 4π.

✓ (b) -4π.

(c) 2π.

(d) -2π.

(e) It depends on the curve bounding the planar piece of surface

Solution Consider for instance the convex envelope of the torus from question 7. It has flat bottom and top parts. Removing a Disk from the flat top and glueing it to the flat piece S we obtain a topological torus. Hence the integral of K on the new surface is 0. However the integral in the convex envelope of the torus is 4π, so the integral of K on S is -4π.