Solutions 2

1. Characterization of convex curves

Let $c \in C^2([0, L], \mathbb{R}^2)$ be a simply C^2-closed curve parametrized by arc-length. Show that the following two statements are equivalent:

(i) The curvature κ_{or} of c doesn't change sign, that is, $\kappa_{\mathrm{or}}(t) \geq 0$ for all $t \in [0, L]$ or $\kappa_{\mathrm{or}} \leq 0$ for all $t \in [0, L]$.

(ii) The curve c is *convex*, that is, the image of c is the boundary of a convex subset $C \subset \mathbb{R}^2$.

Solution.[1] We begin by noticing that c is convex if and only if for each $t \in [0, L]$ the curve lies in one of the closed half-planes determined by the tangent line at $c(t)$.

Let $\dot{c} = e_1 \colon [0, L] \longrightarrow S^1$ be the tangent indicatrix and let $\theta \colon [0, L] \longrightarrow \mathbb{R}$ be a continuous (hence differentiable, as seen in class) polar angle function for e_1, that is,

$$e_1(s) = (\cos\theta(s), \sin\theta(s))$$

for all $s \in [0, L]$. Then

$$\dot{e}_1(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s)) = \theta'(s)e_2(s)$$

and using the first Frenet equation we conclude that $\theta' = \kappa_{\mathrm{or}}$, thus

$$\int_0^t \kappa_{\mathrm{or}}(s)\,\mathrm{d}s = \theta(t) - \theta(0).$$

This shows that the condition that κ_{or} doesn't change sign is equivalent to θ being monotonic.

We now prove $(i) \Rightarrow (ii)$. Suppose that κ_{or} doesn't change sign. Without loss of generality we might assume that it's always ≥ 0 and θ is non-decreasing. Assume that c is not convex. Then there exist $t_0 \in [0, L]$ such that points of $c([0, L])$ can be found on both sides of the tangent line T at $c(t_0)$. Denote by $n \coloneqq \ddot{c}(t_0)/|\ddot{c}(t_0)|$ the normal vector to c at t_0 and define $h \colon [0, L] \longrightarrow \mathbb{R}$ by

$$h(t) \coloneqq \langle c(t) - c(t_0), n\rangle.$$

The map h measures the distance of $c(t)$ from T. Since $[0, L]$ is compact and c runs on both sides of T, the map h has a maximum at $t_1 \neq t_0$ and a minimum at $t_2 \neq t_0$ with $h'(t_1) = h'(t_2) = 0$.

Therefore $\langle \dot{c}(t_0), n\rangle = \langle \dot{c}(t_1), n\rangle = \langle \dot{c}(t_2), n\rangle = 0$, that is, $\dot{c}(t_0), \dot{c}(t_1), \dot{c}(t_2)$ are parallel (but with the three tangent lines at $c(t_0), c(t_1)$ and $c(t_2)$ pairwise distinct) and so there are $s_1 < s_2 \in \{t_0, t_1, t_2\}$ such that $\dot{c}(s_1) = \dot{c}(s_2)$.

Since θ is non-decreasing this shows that $\theta(s_2) - \theta(s_1) = 2\pi k$ for some $k \in \mathbb{N}_0$. The Theorem of turning tangents, together with the fact that θ is non-decreasing, implies that k is either 0 or 1.

If $k = 0$ then θ is constant on $[s_1, s_2]$, which means that $c([s_1, s_2])$ is a straight line parallel to T; this contradicts the fact that the three tangent lines are different.

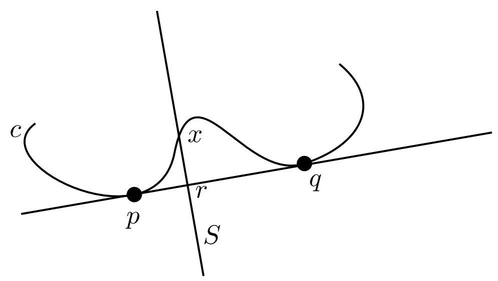
[1] The reader is advised: some pictures might be helpful.

If $k = 1$ then θ must be constant on $[0, s_1]$ and on $[s_2, L]$, which implies that c is a straight line from one of the s_i's and the remaining t_j. Again this is a contradiction.

We now prove $(ii) \Rightarrow (i)$. Suppose that c is convex and that κ_{or} changes sign. Then there are $t_1 < t_2$ in $[0, L]$ with $\theta(t_1) = \theta(t_2)$ and θ not constant on $[t_1, t_2]$. By the Theorem of turning tangents $\dot{c}$ maps surjectively onto S^1, so there exists $t_3 \in [0, L]$ with $\dot{c}t_1 = -\dot{c}(t_3)$.

If the tangent lines at $c(t_1)$, $c(t_2)$ and $c(t_3)$ are pairwise distinct, then they are parallel and one of them lies between the other two. This can't be the case since c is convex, thus two of the tangent lines coincide and there are points $p, q \in \{c(t_1), c(t_2), c(t_3)\}$ lying on the same tangent line.

We claim that the arc of c connecting p to q is the straight line segment $\bar{pq}$ from p to q. Suppose that $r \in \bar{pq}$ is not on c and denote by S the straight line perpendicular to $\bar{pq}$ at r.



Since p and q lie on distinct sides of S, by convexity we know that S is nowhere tangent to c. Thus S intersect c in at least two points, say x and y, where x is the nearest point to r. Then the tangent line to c at x has y on one side and at least one of p and q o the other, contradicting convexity.

Hence r doesn't exist, the arc of c connecting p to q is given by $\bar{pq}$ and p, q must be $c(t_1), c(t_2)$. In paticular θ is constant on $[t_1, t_2]$, contradiction.

2. Submanifolds

Prove that the following matrix groups are submanifolds of $\mathbb{R}^{n\times n}$:

(i) $\mathrm{SL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \ : \ \det A = 1\}$,

(ii) $\mathrm{SO}(n,\mathbb{R}) := \{A \in \mathrm{GL}(n,\mathbb{R}) \ : \ A^{-1} = A^{\mathrm{T}}, \det A = 1\}$.

Solution. The idea is to write $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{R})$ as preimages of regular values of smooth maps.

(i) Consider the smooth map $F\colon \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$, $A \longmapsto \det A$. Notice that F is smooth and $\mathrm{SL}(n,\mathbb{R}) = F^{-1}(1)$ so we want to show that 1 is a regular value for F, that is, $D_AF\colon \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is surjective for all A in $\mathrm{SL}(n,\mathbb{R})$. Since $\mathbb{R}$ is one dimensional it suffices to show that D_AF is not zero for all $A \in \mathrm{SL}(n,\mathbb{R})$. Indeed

$$\begin{aligned} D_AF(A) &= \frac{d}{dt}\Big|_{t=0} F(A+tA) \\ &= \frac{d}{dt}\Big|_{t=0} \det(A(1+t)) \\ &= \frac{d}{dt}\Big|_{t=0} (1+t)^n \det A \\ &= \frac{d}{dt}\Big|_{t=0} (1+t)^n \\ &= n \neq 0. \end{aligned}$$

(ii) Consider the open subset $W := \{A \in \mathbb{R}^{n\times n} \ : \ \det A > 0\} \subset \mathbb{R}^{n\times n}$ and the smooth map $F\colon W \longrightarrow \mathbb{R}^{n(n+1)/2} \cong \mathrm{Symm}(n)$, $A \longrightarrow AA^{\mathrm{T}}$. Notice that

$$\begin{aligned} F^{-1}(I) &= \{A \in \mathbb{R}^{n\times n} \ : \ \det A > 0, AA^{\mathrm{T}} = I\} \\ &= \{A \in \mathbb{R}^{n\times n} \ : \ \det A = 1, AA^{\mathrm{T}} = I\} \\ &= \mathrm{SO}(n,\mathbb{R}). \end{aligned}$$

We want to show that I is a regular value of F, that is, $D_AF\colon \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n(n+1)/2}$ is surjective for all A in $\mathrm{SO}(n,\mathbb{R})$. For $B \in \mathbb{R}^{n\times n}$ we compute

$$\begin{aligned} D_AF(B) &= \frac{d}{dt}\Big|_{t=0} F(A+tB) \\ &= \frac{d}{dt}\Big|_{t=0} (A+tB)(A+tB)^{\mathrm{T}} \\ &= \frac{d}{dt}\Big|_{t=0} (A+tB)(A^{\mathrm{T}}+tB^{\mathrm{T}}) \\ &= \frac{d}{dt}\Big|_{t=0} AA^{\mathrm{T}} + t(BA^{\mathrm{T}} + AB^{\mathrm{T}}) + t^2BB^{\mathrm{T}} \\ &= BA^{\mathrm{T}} + AB^{\mathrm{T}}. \end{aligned}$$

Hence given any X in $\mathbb{R}^{n(n+1)/2} \cong \mathrm{Symm}(n)$, set $B := \frac{1}{2}XA$, then

$$D_AF(B) = \frac{1}{2}XAA^{\mathrm{T}} + A(\frac{1}{2}XA)^{\mathrm{T}} = \frac{1}{2}XAA^{\mathrm{T}} + \frac{1}{2}AA^{\mathrm{T}}X^{\mathrm{T}} = X.$$

This shows that D_AF is surjective for all $A \in F^{-1}(I)$ and $\mathrm{SO}(n,\mathbb{R})$ is a submanifold of $\mathbb{R}^{n\times n}$.

3. Tangent bundle

Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. Show that the *tangent bundle*

$$TM := \bigcup_{p \in M} \{p\} \times TM_p$$

is a $2m$-dimensional submanifold of $\mathbb{R}^{2n}$.

Solution. $TM \subset \mathbb{R}^{2n}$ is a $2m$-submanifold of $\mathbb{R}^{2n}$ if and only if for all $(p_0, X_0) \in TM$ there exist open sets $U \subset \mathbb{R}^{2m}$, $V \subset \mathbb{R}^{2n}$, an immersion $f\colon U \longrightarrow \mathbb{R}^{2n}$ such that $(p_0, X_0) \in f(U) = TM \cap V$ and $f\colon U \longrightarrow TM \cap V$ is a homeomorphism.

Let $(p_0, X_0) \in TM$ then there exists a local parametrization: open sets $U_0 \subset \mathbb{R}^m$, $V_0 \subset Rn$, an immersion $\varphi\colon U_0 \longrightarrow \mathbb{R}^n$ with $p_0 \in \varphi(U_0) = M \cap V_0$ and $\varphi\colon U_0 \longrightarrow M \cap V_0$ is a homeomorphism.

We define a local parametrization of TM at (p_0, X_0) as follows: let $U := U_0 \times \mathbb{R}^m \subset \mathbb{R}^{2m}$, $V := V_0 \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ and $f\colon U \longrightarrow \mathbb{R}^{2n}$ defined by

$$f(x, \xi) = (\varphi(x), D_x\varphi(\xi)).$$

Since φ is smooth, the same holds for f. The derivative of f at (x, ξ) is given by

$$D_{(x,\xi)} f = \begin{pmatrix} D_x\varphi & 0 \\ * & D_x\varphi \end{pmatrix},$$

because $D_\xi(D_x\varphi) = D_x\varphi$ by linearity[2] of $D_x\varphi$. As φ is an immersion $D_x\varphi$ has rank m, so $D_{x,\xi} f$ has rank $2m$ and we conclude that f is an immersion.

Moreover

$$\begin{aligned} f(U) = f(U_0 \times \mathbb{R}^m) &= \bigcup_{x \in U_0} \{\varphi(x)\} \times D_x\varphi(\mathbb{R}^m) = \bigcup_{p \in \varphi(U_0)} \{p\} \times TM_p \\ &= \bigcup_{p \in M \cap V_0} \{p\} \times TM_p = TM \cap (V_0 \times \mathbb{R}^n) = TM \cap V \end{aligned}$$

and $(p_0, X_0) \in f(U)$.

In order to show that $f\colon U \longrightarrow TM \cap V$ is a homeomorphism, consider $g\colon TM \cap V \longrightarrow U$:

$$g(p, X) := \big(\varphi^{-1}(p), D_p\varphi^{-1}(X)\big).$$

The functions φ^{-1} and $D_p\varphi^{-1}$ are continuous and so is g. Moreover

$$\begin{aligned} f \circ g(p, X) &= f\big(\varphi^{-1}(p), D_p\varphi^{-1}(X)\big) \\ &= \big(p, D_{\varphi^{-1}(p)}\varphi D_p\varphi^{-1}(X)\big) \\ &= \big(p, D_p(\varphi \circ \varphi^{-1})(X)\big) \\ &= \big(p, X\big), \end{aligned}$$

and similarly

$$g \circ f(x, \xi) = g\big(\varphi(x), D_x\varphi(\xi)\big) = \big(\varphi^{-1} \circ \varphi(x), D_{\varphi(x)}\varphi^{-1} D_x\varphi(\xi)\big) = (x, \xi).$$

This shows that $f^{-1} = g$ is continuous.

[2]

$$D_\xi(D_x\varphi)(X) = \frac{d}{dt}\Big|_{t=0} D_x\varphi(\xi + tX) = \frac{d}{dt}\Big|_{t=0} D_x\varphi(\xi) + tD_x\varphi(X) = D_x\varphi(X).$$