Solutions 8

1. Chebyshev Nets

Let $f\colon U \to \mathbb{R}^3$ be a parametrized surface with $U = (0, A) \times (0, B) \subset \mathbb{R}^2$.

a) Show that the following two conditions are equivalent:

(i) For every rectangle $R = [u_1, u_1 + a] \times [u_2, u_2 + b] \subset U$ the opposite sides of $f(R)$ have the same length.

(ii) $\frac{\partial g_{11}}{\partial u_2} = \frac{\partial g_{22}}{\partial u_1} \equiv 0$ on U.

If f satisfies one of the equivalent conditions, then its parameter lines constitute a *Chebyshev net*.

b) Show that for such a parametrization there exists a change of coordinates $\varphi\colon U \to \tilde{U}$ such that the first fundamental form of $\tilde{f} := f \circ \varphi^{-1}$ has the form

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix},$$

where ω is the angle between the parameter lines of f.

Solution. a) For a rectangle $R = [u_1, u_1 + a] \times [u_2, u_2 + b] \subset U$ we consider the two opposite boundary curves $\gamma, \tilde{\gamma}\colon [0, a] \to U$:

$$\gamma(t) := (u_1 + t, u_2),$$
$$\tilde{\gamma}(t) := (u_1 + t, u_2 + b).$$

As $\dot{\gamma}(t) = \dot{\tilde{\gamma}}(t) = e_1$, it holds that the length of γ and $\tilde{\gamma}$ satisfy $L(\gamma) = \int_0^a \sqrt{g_{11}(u_1 + t, u_2)}\,\mathrm{d}t$ and $L(\tilde{\gamma}) = \int_0^a \sqrt{g_{11}(u_1 + t, u_2 + b)}\,\mathrm{d}t$.

Now, if $L(\gamma) = L(\tilde{\gamma})(= L(f \circ \gamma) = L(f \circ \tilde{\gamma}))$ then

$$\int_0^a \sqrt{g_{11}(u_1 + t, u_2)}\,\mathrm{d}t = \int_0^a \sqrt{g_{11}(u_1 + t, u_2 + b)}\,\mathrm{d}t$$

and as this holds for all $a \in [0, A - u_1]$ we can differentiate with respect to a and obtain

$$g_{11}(u_1 + t, u_2) = g_{11}(u_1 + t, u_2 + b).$$

Therefore g_{11} is constant along the u_2-Axis, so $\frac{\partial g_{11}}{\partial u_2} = 0$. Analogously we obtain $\frac{\partial g_{22}}{\partial u_1} = 0$.

On the other hand if $\frac{\partial g_{11}}{\partial u_2} = 0$, then $g_{11}(u_1 + t, u_2) = g_{11}(u_1 + t, u_2 + b)$ for all b and therefore $L(\gamma) = L(\tilde{\gamma})$. Analogously for the other two sides.

b) Define $\phi(u_1, u_2) \coloneqq (w_1(u_1), w_2(u_2))$, where

$$w_1(x) \coloneqq \int_0^x \sqrt{g_{11}(t, u_2)}\,\mathrm{d}t,$$
$$w_2(y) \coloneqq \int_0^y \sqrt{g_{22}(u_1, t)}\,\mathrm{d}t.$$

Notice that because of (ii) w_1 and w_2 are independent from the choice of u_2 and u_1, respectively. Then

$$d\phi = \begin{pmatrix} \sqrt{g_{11}} & 0 \\ 0 & \sqrt{g_{22}} \end{pmatrix}.$$

Hence ϕ and $d\phi$ are injective and ϕ is a diffeomorphism onto its image $\tilde{U} \coloneqq \phi(U)$. Now,

$$(d\phi)^{-1} = \begin{pmatrix} \frac{1}{\sqrt{g_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{g_{22}}} \end{pmatrix}$$

and hence

$$\tilde{g}_{ij} = \tilde{g}(e_i, e_j) = g\left(d\phi^{-1}(e_i), d\phi^{-1}(e_j)\right) = g\left(\frac{1}{\sqrt{g_{ii}}}e_i, \frac{1}{\sqrt{g_{jj}}}e_j\right) = \frac{g_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}},$$

so $\tilde{g}_{11} = \tilde{g}_{22} = 1$ and $\tilde{g}_{12} = \frac{\langle f_1, f_2 \rangle}{|f_1||f_2|} = \cos\omega$.

2. Isoperimetric problem on a Cartan-Hadamard surface

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface, such that f is a homeomorphism between $\mathbb{R}^2$ and $M := f(\mathbb{R}^2)$. Assume that f has nonnegative Gauss curvature K. Given $\Omega \subset M$ bounded, we say that $\partial\Omega$ is C^2 if it consists of a finite disjoint union of C^2 simple closed curves. For such Ω define the *isoperimetric quotient*

$$\mathcal{I}(\Omega) := \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)^{\frac{1}{2}}}$$

a) Suppose first that M is isometric to the Euclidean plane. Show that if Ω_0 is a minimizer of $\mathcal{I}$ (such that $\partial\Omega_0$ is C^2) then

$$\mathcal{I}(\Omega_0) = \sqrt{4\pi} \quad \text{and } \Omega_0 \text{ is an Euclidean disc.}$$

Hint: Show that, by minimality, $\partial\Omega_0$ must consist of only one closed simple curve γ, and prove (using the first variation of arc length) that the geodesic curvature κ_g of γ must be constant. Deduce that γ must trace a circle in $\mathbb{R}^2$.

b) For general $K \le 0$, show that if Ω_0 is a minimizer of $\mathcal{I}$ (with $\partial\Omega_0$ of class C^2) then it must be $K \equiv 0$ in Ω_0.

Hint: Using $\Omega_r = f(B_r(0))$, with $r \to 0^+$ as competitors, show that $\mathcal{I}(\Omega_0) \le \sqrt{4\pi}$. Show that, as in a), $\partial\Omega_0$ must consist of only one closed simple curve γ. Let ν be the inwards unit normal to $\partial\Omega_0$, define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} I(\Omega_\varepsilon) \le 0$, and < 0 unless $K \equiv 0$ in Ω_0.

Solution. a) We can assume without loss of generality $M = \mathbb{R}^2$, since the isoperimetric problem is intrinsic. Note that if Ω_0 has multiple components each is a closed simple curve. Hence, the image of each of these curves it divides $\mathbb{R}^2$ into two connected components (one bounded and one unbounded). Now, the union of (the closures of) the bounded components is a new set which contains Ω_0 and whose boundary is contained in $\partial\Omega_0$. Hence, this set obtained by "filling the holes" it would have more area and less perimeter, contradicting the fact that Ω_0 minimizes $\mathcal{I}$.

Let $\gamma : (0, L) \to \mathbb{R}^2$ be a curve tracing $\partial\Omega_0$, parametrized by the arc length, and let $\nu : [0, L] \to \mathbb{S}^1$ be the inwards unit normal. Given $\xi \in C^2_{\text{closed}}([a, b])$ define $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\xi(t)\nu(t)$ and let Ω_ε be the bounded connected component of $\mathbb{R}^2 \setminus \text{image}(\gamma_\varepsilon)$. If $\int_0^L \xi(t) = 0$ then $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \text{area}(\Omega_\varepsilon) = 0$. Hence be minimality it must be $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \text{length}(\Omega_\varepsilon) = \int_0^L \kappa_g(t)\xi(t)dt = 0$. Since ξ is an arbitrary average zero smooth function we deduce that $\kappa_g \equiv \kappa = \textit{constant}$ or equivalently $c'' \equiv \kappa\nu$. This easily implies that c traces a circle with radius $1/\kappa$.

b) Reparametrize $f(\mathbb{R}^2)$ as $\tilde{f}(x) = f(Ax)$, where A is a matrix with nonzero determinant, so that $g_0 = \text{Id}$, so $|g_x - \text{Id}| \le Cr$ for $|x| < r$. Then, $\text{area}(\tilde{f}(B_r(0))) = r^2(1 + O(r))$ and $\text{length}(\partial B_r) = 2\pi r(1 + O(r))$. It follows that

$$\mathcal{I}(\tilde{f}(B_r(0)) = \sqrt{4\pi} + o(1) \quad \text{as } r \downarrow 0.$$

Since by assumption Ω_0 is a minimizer of $\mathcal{I}$ it must be $\mathcal{I}(\Omega_0) \le \mathcal{I}(\tilde{f}(B_r(0))$ for all $r > 0$ and hence $\mathcal{I}(\Omega_0) \le \sqrt{4\pi}$.

As in a) —now using that M is homeomophic to $\mathbb{R}^2$—, $\partial\Omega_0$ must consist of only one simple closed curve $f \circ \gamma$. Let us take γ oriented counterclockwise and let ν be the inwards unit normal to $\partial\Omega_0$ (as in the Gauss-Bonet setting). Define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Let us show that show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \mathcal{I}(\Omega_\varepsilon) \le 0$, and < 0 unless $K \equiv 0$ in Ω_0.

Indeed, on the one hand $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \text{area}(\Omega_\varepsilon) = -\text{length}(\partial\Omega_0)$. On the other hand, $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \text{length}(\Omega_\varepsilon) = -\int_{\partial\Omega_0} \kappa_g ds$

Now, using Gauss-Bonnet, $\int_{\partial\Omega_0} \kappa_g ds = 2\pi - \int_{\Omega_0} K dA \geq 2\pi$ ($>2\pi$ unless $K \equiv 0$). Hence,

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{I}(\Omega_\varepsilon) &= \frac{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\text{length}(\partial\Omega_\varepsilon)}{\text{area}(\Omega_0)^{\frac{1}{2}}} - \frac{1}{2}\frac{\text{length}(\partial\Omega_0)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\text{area}(\Omega_\varepsilon)}{\text{area}(\Omega_0)^{\frac{3}{2}}} \\
&\leq (<) - \frac{2\pi}{\text{area}(\Omega_0)^{\frac{1}{2}}} + \frac{1}{2}\frac{\text{length}(\partial\Omega_0)^2}{\text{area}(\Omega_0)^{\frac{3}{2}}} \\
&= -\frac{2\pi}{\text{area}(\Omega_0)^{\frac{1}{2}}} + \frac{1}{2}\frac{\mathcal{I}(\partial\Omega_0)^2}{\text{area}(\Omega_0)^{\frac{1}{2}}} \leq 0,
\end{aligned}$$

since $\mathcal{I}(\Omega_0)^2 \leq 4\pi$. This contradicts the minimality of Ω_0 unless the second inequality is an equality, which implies that $K \equiv 0$ in Ω_0.

3. The Brouwer Fixed Point Theorem

The Brouwer fixed point theorem states:

Theorem. *Let $D := \{x \in \mathbb{R}^n \ : \ |x| \leq 1\}$ be the unit ball. Then every continuous map $f\colon D \to D$ has a fixed point.*

a) Let $M \subset \mathbb{R}^3$ be a surface and $\tilde{D} \subset M$ a region diffeomorphic to the disc $D := \{x \in \mathbb{R}^2 \ : \ |x| \leq 1\}$. Consider a tangent vector field $X\colon \tilde{D} \to \mathbb{R}^3$ which on $\partial\tilde{D}$ is pointing outward. Show that X has zeros in the interior of $\tilde{D}$.

b) Prove the Brouwer fixed point theorem in two dimensions using part a).

Solution. a) It suffices to prove the statement for $X\colon D \to \mathbb{R}^2$.

We want to use Poincaré index theorem, but for that we must have a compact surface without boundary.

First we can modify X such that on ∂D it points radially towards the exterior. Then we consider $Y\colon D \to \mathbb{R}^2$, $Y := -X$, which is a vector field on D pointing radially towards the interior on every point of ∂D.

Now identify D with an hemisphere of S^2, then we can glue two hemisphere together along their boundaries to obtain S^2. By considering the vector field X on one hemisphere and Y on the other we obtain a continuous vector field $Z\colon S^2 \to \mathbb{R}^3$, which is nowhere vanishing on the equator. As seen in class, the Poincaré index theorem implies that Z must have a zero, but since there are non on the equator we conclude that X or $-X$ (and hence X) must have at least one singularity in the interior of D.

b) Let $f\colon D \to D$ be a continuous map. We define the vector field $X\colon D \to \mathbb{R}^2$ by $X(x) \coloneqq x - f(x)$. For $x \in \partial D$ it holds

$$\langle X(x), x\rangle = \langle x, x\rangle - \langle f(x), x\rangle \geq 1 - |x||f(x)| \geq 0.$$

This shows that if X doesn't vanish on ∂D, then it points outward everywhere. In this case it follows from a) that it has a zero in the interior of D. In both cases there is x_0 with $X(x_0) = 0$, that is $f(x_0) = x_0$.