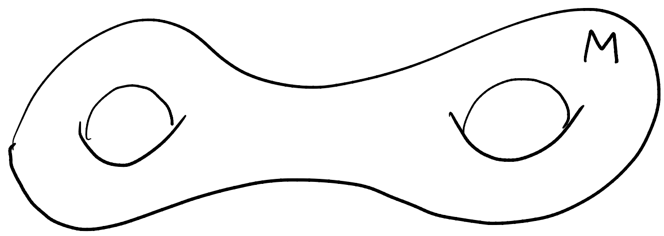
Single Choice Questions

SC1 Let M be the (genus 2) surface in the figure.



Then $\int_M K\,dA$ equals:

(A) 4π.

(B) 0.

(C) -2π.

(D) -4π.

(E) -8π.

SC2 Let M be the same surface as in question SC1. If $N\colon M \longrightarrow \mathbb{S}^2$ is the Gauss map, then $\deg_2 N$, i.e. the mapping degree mod 2, equals

(A) 0.

(B) 1.

(C) depends on whether N is pointing inwards or outwards.

(D) 1 at points of positive Gauss curvature, 0 otherwise.

(E) 0 at points of positive Gauss curvature, 1 otherwise.

SC3 Consider the following (graphical) submanifolds of $\mathbb{R}^3$, parametrized by $f\colon (x,y) \mapsto (x, y, z(x,y))$, where $z = z(x,y)$ is given by

(I) $z = x^2 + y^2$ (II) $z = x^2 - y^2$ (III) $z = x^4 - y^2$

(IV) $z = x^2 - \sin^2 y$ (V) $z = y^2 + (x+y)^3$

The following is true.

(A) (I) and (V) have positive Gauss curvature at $(0,0)$.

(B) (II), (III), and (IV) have negative Gauss curvature at $(0,0)$.

(C) (III) and (V) have zero Gauss curvature at $(0,0)$.

(D) (II) is the only one with negative Gauss curvature at $(0,0)$.

(E) (V) has positive Gauss curvature at $(0,0)$.

SC4 Suppose $\Omega \subset \mathbb{R}^3$ is an open subset such that $\partial\Omega$ (the topological boundary) is a smooth compact submanifold satisfying $\int_{\partial\Omega} |K|\, dA = 4\pi$. Then

(A) Ω must be convex.

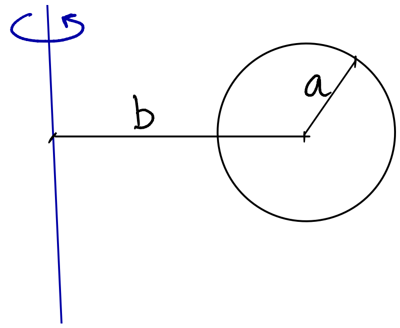
(B) Ω must be a ball.

(C) Ω must be a union of disjoint balls.

(D) Ω must have constant mean curvature.

(E) the mean curvature of Ω must vanish at one point, at least.

SC5 Consider the torus of revolution as in the figure.



Let Ω be the set of points with positive Gauss curvature. The area of Ω is:

(A) $2\pi^2 ab$.

(B) $2\pi^2 ab + 4\pi a^2$.

(C) $4\pi ab$.

(D) $2\pi(a^2 + b^2)$.

(E) $\pi^2 a^2 + \pi ab$.

SC6 For Ω as in question SC5, the integral $\int_\Omega K\, dA$ equals:

(A) 4π.

(B) $\pi^2 \dfrac{b-a}{a+b}$.

(C) $\dfrac{8\pi ba}{a^2+b^2}$.

(D) $2\pi\Big(1 + \dfrac{2ab}{a^2+b^2}\Big)$.

(E) $4\pi \dfrac{a^2}{a^2+b^2}$.

Problem 1
The sphere revisited

Consider the parametrization f given by the inverse of the stereographic projection $\mathbb{S}^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$ from the North pole. Explicitly, $f\colon \mathbb{R}^2 \longrightarrow \mathbb{S}^2 \setminus \{(0,0,1)\}$ is given by

$$f\colon (x,y) \longmapsto \Big(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\Big).$$

(a) Compute the first fundamental form of f. Is f conformal?

Consider $\mathbb{R}^2$, the domain of the parametrization f. The *metric distance* between two points (x,y) and (x',y') is defined as

$$\inf\Big\{L(\gamma)\;\Big|\;\gamma\colon [0,1] \to \mathbb{R}^2 \text{ piecewise smooth curve with } \gamma(0) = (x,y), \gamma(1) = (x',y')\Big\},$$

where the length L of the curve γ is defined as $L(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$, and g is the first fundamental form of the parametrization f.

(b) Show that for any parallel, represented on $\mathbb{R}^2$ by a circle $x^2+y^2=r^2$ for some $r>0$, the *metric distance* between $(0,0)$ and any of its points (x_0,y_0), depends only on r and is attained by the curve $s \longmapsto s(x_0,y_0)$, defined for $s \in [0,1]$.

Hint: For $a \in \mathbb{R}$, $\int \frac{1}{a^2+s^2}\, ds = \frac{1}{a}\arctan(\frac{s}{a}) + c.$

(c) Prove that the distance between any two points on the sphere (which are not antipodal) is attained by the great circular arc joining them.

Hint: you can either deduce it using (b) or give another proof, e.g. the one that was given in the first lecture.

Problem 2

Local isothermal coordinates on a minimal surface

(a) Suppose that U is a ball in the (x, y)-plane $\mathbb{R}^2$. Show that, given smooth functions $A, B\colon U \longrightarrow \mathbb{R}$, the equations $\frac{\partial}{\partial x}\Phi = A$, $\frac{\partial}{\partial y}\Phi = B$ admit a solution Φ if and only if $\frac{\partial}{\partial y}A = \frac{\partial}{\partial x}B$ holds.

(b) Let $U \subset \mathbb{R}^2$ be a ball and let $v\colon U \longrightarrow \mathbb{R}$ be such that $f\colon U \longrightarrow \mathbb{R}^3$,

$$f(x, y) \coloneqq (x, y, v(x, y))$$

is a minimal immersion (i.e. the mean curvature or the trace of the Weingarten map vanish for all (x, y)). Show that v must satisfy the PDE

$$\frac{\partial}{\partial x}\Big(\frac{v_x}{W}\Big) + \frac{\partial}{\partial y}\Big(\frac{v_y}{W}\Big) = 0, \qquad (\circledast)$$

where $W = W(x, y) \coloneqq \sqrt{1 + v_x^2 + v_y^2}$, and where the subindexes x and y denote partial derivatives.

(c) Show that equation $(\circledast)$ from part (b) implies

$$\frac{\partial}{\partial x}\Big(\frac{1 + v_y^2}{W}\Big) = \frac{\partial}{\partial y}\Big(\frac{v_x v_y}{W}\Big),$$

and deduce the existence of a potential Φ for the vector field $(\frac{v_x v_y}{W}, \frac{1+v_y^2}{W})$, that is, a function $\Phi\colon U \longrightarrow \mathbb{R}$ such that

$$\frac{\partial}{\partial x}\Phi = \frac{v_x v_y}{W} \qquad\qquad \frac{\partial}{\partial y}\Phi = \frac{1 + v_y^2}{W}.$$

(d) Introduce new coordinates $\bar{x} = x$, $\bar{y} = \Phi(x, y)$ and check that the first fundamental form with respect to the new coordinates is of the form

$$\bar{g}_{(\bar{x},\bar{y})} = \lambda(\bar{x}, \bar{y}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Compute the conformal factor $\lambda(\bar{x}, \bar{y})$.

Hint: put $(\bar{x}, \bar{y}) = \Psi(x, y) \coloneqq (x, \Phi(x, y))$ and compute the first fundamental form of the parametrization $f \circ \Psi^{-1}\colon \Psi(U) \longrightarrow f(U)$, $(\bar{x}, \bar{y}) \longmapsto f(\Psi^{-1}(\bar{x}, \bar{y}))$, using the chain rule.

(e) Write the statement and proof of a theorem given in the lecture concerning the harmonicity of the coordinate functions for an isothermal parametrization of a minimal surface.