# Exam Problems HS21

## 1. The Catenoid

Consider the following parametrization of a "cylindrical" surface of revolution in $\mathbb{R}^3$:

$$S_f \coloneqq \{x \in \mathbb{R}^3 \mid \sqrt{(x^1)^2 + (x^2)^2} = f(x^3),\ |x^3| \leq 1\},$$

where $f\colon [-1,1] \to (0,\infty)$ is smooth. Notice that $\partial S_f$ is the union of two circumferences.

(a) Show that the area of $S_f$ is given by $2\pi \int_{-1}^{1} f(t)\sqrt{1 + f'(t)^2}\, dt$.

(b) Prove that if $S_f$ has minimal area among all cylindrical surfaces as above with the same boundary, then $f$ must satisfy:

(i) $ff'' = 1 + (f')^2$,

(ii) $\left(\frac{f}{\sqrt{1+(f')^2}}\right)' = 0$.

(c) Show that solutions of (ii) must be of the form $f(t) = a\cosh(\frac{t-b}{a})$ for some $b \in \mathbb{R}$ and $a > 0$.
[*Hint:* Use $\int \frac{dy}{\sqrt{y^2-a^2}} = \cosh^{-1}(\frac{y}{a}) + \text{constant}$.]

(d) Prove that $\int_{S_f} K\, dA = -2\pi \int_{f'(-1)}^{f'(1)} \frac{dz}{(1+z^2)^{3/2}}$.

## 2. Lie Bracket and Curvature

Let $X$ be a $C^\infty$ vector field on an open set $U \subset \mathbb{R}^n$. By the identification of vector fields and derivations, $X$ acts on $C^\infty$ functions:

$$Xf = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j} f \quad \text{for } f \in C^\infty(U),$$

where $X^i$ are the components[1] of $X$. Similarly, if $Y$ is a another $C^\infty$ vector field on $U$, we let $X$ act on $Y$ component-wise. That is, we denote $XY$ the vector field with components

$$(XY)^i \coloneqq XY^i = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j} Y^i.$$

[1] That is, $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$ in the derivation point of view, or simply $X(p) = (X^1(p), X^2(p), \ldots, X^n(p)) \in \mathbb{R}^n \cong TU_p$ for all $p \in \mathbb{R}^n$.

Given two smooth vector fields $X, Y$ as above, define their *Lie bracket* $[X, Y] \coloneqq Z$ as the map $Z\colon C^\infty(U) \to C^\infty(U)$ defined by

$$Zf \coloneqq X(Yf) - Y(Xf).$$

(a) Show that $Zf$ can be written as $\sum Z^i \frac{\partial}{\partial x^i} f$ and compute $Z^i$ in terms of the $X^i$'s and $Y^i$'s. Deduce that $Z$ is a smooth vector field on $U$.

(b) Prove $[X, Y] = XY - YX$.

(c) Let $\psi\colon U \to V$ be a $C^\infty$ diffeomorphism. Define the *push-forward* of a vector field $T$ on $U$, denoted $\psi_* T$, as

$$\psi_*(T)(q) \coloneqq d\psi_{\psi^{-1}(q)} T(\psi^{-1}(q)),$$

for $q \in V$. Prove that $[\psi_*(X), \psi_*(Y)] = \psi_*([X, Y])$.
[*Hint:* Use that for all smooth $f\colon V \to \mathbb{R}$, and vector field $T$ on $U$ we have $(\psi_*(T)f) \circ \psi = T(f \circ \psi)$.]

(d) Show that if $M \subset \mathbb{R}^3$ is an embedded surface, and $X, Y$ are tangent vector fields on $M$, then the Lie bracket $[X, Y]$ is well-defined as $[\tilde{X}, \tilde{Y}]$, where $\tilde{X}$, $\tilde{Y}$ are extensions of $X, Y$ to an open set $U \subset \mathbb{R}^3$ containing $M$. Prove that $[X, Y]$ is also tangent to $M$.
[*Hint:* You may assume that such extensions always exists. Also, if $\psi\colon W \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^3$, then $T$ is tangent to $M \cap W$ if and only if $\psi_*(T)$ is tangent to $\psi(M \cap W) \subset V$.]

(e) With $M$ as above, show that if $X, Y$ are tangent vector fields such that $X(p)$, $Y(p)$ is an orthonormal basis of $TM_p$ for all $p \in M$, then

$$\langle D_X D_Y X - D_Y D_X X - D_{[X,Y]} X, Y\rangle = -K,$$

where $D$ denotes the covariant derivative and $K$ is the Gauss curvature.
[*Hint:* Denoting $\nu$ the unit normal to $M$, recall that

$$K = \langle XX, \nu\rangle\langle YY, \nu\rangle - \langle YX, \nu\rangle\langle XY, \nu\rangle.$$

Using the previous expression for $K$, and that $X, Y, \nu$ are orthonormal, prove $-K = \langle Y(\langle XX, \nu\rangle\nu) - X(\langle YX, \nu\rangle\nu), Y\rangle$.

Also, recall the definition of covariant derivative $D_Z T = ZT - \langle ZT, \nu\rangle\nu$ for any tangent vector fields $Z, T$.]

### 3. Sard's Lemma and Whitney's Embedding Theorem

Let $M$ be a compact $m$-dimensional $C^\infty$ manifold. Recall that there exists an embedding $F\colon M \to \mathbb{R}^n$ for a (possibly very large) $n$ depending on $M$ (Theorem 8.9 in the lecture). The goal of this problem is to lower the dimension $n$ to $2m+1$.

(a) Let $\tilde{M} \subset \mathbb{R}^n$ be a compact $m$-dimensional $C^\infty$ submanifold. Prove that

$$\mathcal{U}T\tilde{M} := \big\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \ : x \in \tilde{M}, \xi \in T\tilde{M}_x, |\xi| = 1\big\}$$

is a $(2m-1)$-dimensional compact $C^\infty$ submanifold of $\mathbb{R}^{2n}$.
[*Hint:* Using a submanifold chart $(\psi, U)$ notice that $\tilde{M} \cap U$ can be written as $\{x \in U : \psi^{m+1}(x) = \cdots = \psi^n(x) = 0\}$. Try to write $\mathcal{U}T\tilde{M}$ locally as the zero set of a certain map $G\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n-2m+1}$ having 0 as a regular value.]

(b) Given $e \in \mathbb{S}^{n-1}$ define

$$e^\perp := \{x \in \mathbb{R}^n : \ e \cdot x = 0\} \cong \mathbb{R}^{n-1},$$

and let $\pi_e\colon \mathbb{R}^n \to e^\perp$ be the orthogonal projection $x \mapsto x - (e \cdot x)e$. Prove that $\pi_e|_{\tilde{M}}$ is an immersion if and only if $e$ does *not* belong to the image of the map $\pi_2|_{\mathcal{U}T\tilde{M}}\colon \mathcal{U}T\tilde{M} \to \mathbb{S}^{n-1}$, defined as the restriction of the canonical projection $\pi_2(x,\xi) = \xi$.

(c) Prove that $\pi_e|_{\tilde{M}}$ is injective if and only if $\pm e$ do *not* belong to the image of the map $g\colon (\tilde{M} \times \tilde{M}) \setminus \Delta \to \mathbb{S}^{n-1}$, defined as

$$g(x,y) := \frac{x-y}{|x-y|},$$

where $\Delta := \{(x,x)\colon x \in \tilde{M}\}$.

(d) Using Sard's Lemma, show that if $n > 2m+1$, then for almost every $e \in \mathbb{S}^{n-1}$ the projection $\pi_e\colon \tilde{M} \to e^\perp$ is an injective immersion.
[*Hint:* Recall Sard's Lemma. If $F\colon M^m \to N^n$ is a $C^r$ map with $r > \max\{0, n-m\}$, then the set of singular values of $F$ has measure zero in $N$. ]

(e) Prove Whitney's Embedding Theorem (compact case), namely, that every smooth compact $m$-dimensional manifold can be embedded in $\mathbb{R}^{2m+1}$.

# Differential Geometry I exam (multiple choice part)

**1.** The radius of the osculating circle of the curve $c(t) := (at, -t^2)$, $a > 0$, at the point $(0,0)$ is given by:

(a) $\frac{a^2}{2}$.

(b) $\frac{a}{2}$.

(c) 2.

(d) 1.

(e) $\frac{\sqrt{a}}{2}$.

**2.** Assume that a (smooth, nonempty) compact 2-dimensional submanifold $M \subset \mathbb{R}^3$ satisfies $\int_M H^2 \, dA = \int_M K \, dA$, where $K$ is the Gauss curvature and $H$ is the mean curvature[1]. Then $M$ must be a/an

(a) sphere.

(b) union of spheres.

(c) ellipsoid.

(d) Clifford torus.

(e) point.

**3.** Consider the differential 1-form $\omega = -ydx + xdy$ in $\mathbb{R}^2$. Let $D$ be the ellipsoid $\{(x,y) : ax^2 + \frac{1}{a}y^2 \leq 1\}$, where $a > 0$. Then $\int_{\partial D} \omega$ equals:

(a) $2\pi$.

(b) $\pi\big(a + \frac{1}{a}\big)$.

(c) $e^{2\pi i}$.

(d) $-\pi\cos(\mathrm{a})$.

(e) 1.

[1] The average of the principal curvatures.

**4.** Let $C \subset \mathbb{R}^3$ be the cylinder in $\mathbb{R}^3$, parametrized as

$$f(u,v) = \big(R\cos u, R\sin u, v^3\big),$$

$R > 0$. What are the correct values of the Gauss curvature $K$ and mean curvature $H$ at the point $(R, 0, \sqrt[3]{2}) \in C$ (with respect to the outward pointing Gauss map)?

(a) $K = 0, H = 0.$

(b) $K = 0, H = -\frac{1}{2R}.$

(c) $K = 0, H = \frac{1}{2R} + 1.$

(d) $K = \frac{2}{R^2}, H = \frac{1}{R}.$

(e) $K = 0, H = -R/2.$

**5.** Consider a "spherical pentagon" (geodesically convex region bounded by five circular arcs) of area $A$ in a 2-sphere of $\mathbb{R}^3$ with area $A'$. The sum of its (five) interior angles is:

(a) $\pi\big(3 + 4\frac{A}{A'}\big).$

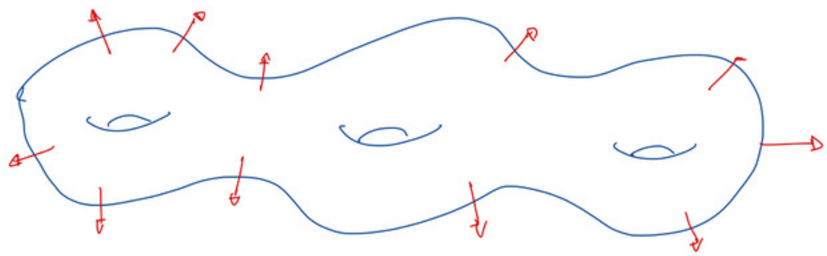
(b) $2\pi + A/A'.$

(c) $3\pi + A.$

(d) $5\pi + 2A/\sqrt{A'}.$

(e) $5\pi + A/\sqrt{A'}$

**6.** Let $M \subset \mathbb{R}^3$ be the smooth surface as depicted:



What is the value of the integral of the Gauss curvature $K$ over $M$ (with respect to the differential of the area)?

(a) $-3\pi$.

(b) 0.

(c) depends on how $M$ is embedded in $\mathbb{R}^3$.

(d) $-6\pi$.

(e) $-8\pi$.

**7.** Which one is true?

(a) Cylinders, spheres and planes are the only connected submanifolds of $\mathbb{R}^3$ with constant mean curvature.

(b) A smooth compact surface in $\mathbb{R}^3$ whose area is minimal among all surfaces enclosing the same volume must have constant mean curvature.

(c) Any connected embedded minimal surface $M \subset \mathbb{R}^3$ with $\int_M K\,dA > -8\pi$ must be a plane.

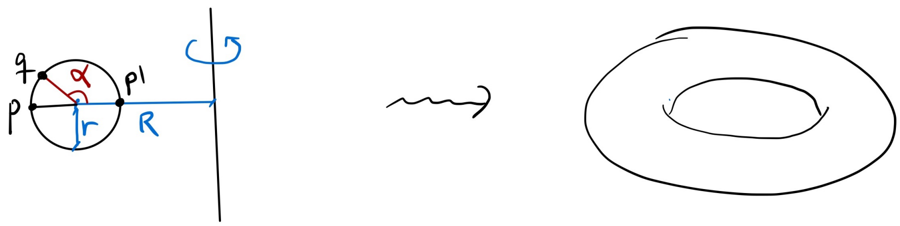
(d) Alexandroff's theorem classifies all constant mean curvature embedded surfaces.

(e) If a constant mean curvature surface is embedded and non-compact, then it must be a cylinder.

**8.** Consider the torus of revolution

$$f(x,y) = \big(\cos x(-R + r\cos y), \sin x(-R + r\cos y), r\sin y\big),$$

$R > r > 0$, drawn below:




Its mean curvature (with respect to the outward pointing Gauss map) at $p = (-R-r, 0, 0)$ is:

(a) $-\frac{1}{2}\big(\frac{1}{r} + \frac{1}{\sqrt{R^2-r^2}}\big)$.

(b) $-\frac{1}{2}\big(\frac{1}{\sqrt{rR}} + \frac{1}{R-r}\big)$.

(c) $-\frac{1}{2}\big(\frac{1}{r} + \frac{1}{R+r}\big)$.

(d) $-\frac{1}{2}\big(\frac{1}{r} - \frac{1}{R+r}\big)$.

(e) $-\frac{1}{2}\big(\frac{1}{r} - \frac{1}{\sqrt{R^2-r^2}}\big)$.

**9.** Consider again the torus from the previous question. At any point of the torus one principal curvature is $-1/r$. The other principal curvature at the point $q = (-R + r\cos\alpha, 0, r\sin\alpha)$ is:

(a) $\frac{\cos\alpha}{R-r\cos\alpha}$.

(b) $\frac{\cos\alpha}{R-r}$.

(c) $\frac{\tan\alpha}{R-r}$.

(d) $\frac{\cos\alpha}{R-r\sin\alpha}$.

(e) $\frac{\tan\alpha}{R+r}$.

**10.** Consider again the torus from the previous two questions. When the point $q$ is rotated about the $x_3$-axis, it generates the curve $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$, which is contained in the torus. Given a tangent vector $X$ at $q$ consider its parallel transport along $\gamma$ for one full turn ($t \in [0, 2\pi]$), producing a new tangent vector $Y$ at $q$. The angle between $X$ and $Y$ is:

(a) $\frac{\alpha R}{r}$.

(b) $2\pi \sin\alpha$.

(c) $\frac{\tan\alpha R}{r}$.

(d) $2\pi\cos\alpha$.

(e) $\sin\alpha$.