Repetition Exam - Open Problems

1. Gauss-Bonnet and moving frames

Let $U \subset \mathbb{R}^2$ be open, $f : U \to \mathbb{R}^3$ an embedding and let $M = f(U)$.

a) Show that there exist two smooth vector fields $E_i : M \to \mathbb{R}^3$, $i = 1, 2$, which are tangent to M and satisfy $E_1 \circ f = \frac{\partial_1 f}{|\partial_1 f|}$ and $\langle E_i, E_j \rangle = \delta_{ij}$.

b) Suppose that $\bar{D} \subset M$ is homeomorphic to a disk and is bounded by a smooth unit speed curve $c : [0, L] \to M$. Let $\nu(s)$ be the unit normal to $\bar{D}$ at $c(s)$ pointing towards the interior of $\bar{D}$, and suppose that $c'(s), \nu(s)$ has the same orientation as E_1, E_2. Prove that

$$\int_D K dA = -\int_0^L \langle (E_1 \circ c)', E_2 \circ c \rangle ds,$$

where K is the Gauss curvature.

Hint: Consider a continuous angle $\varphi : [0, L] \to \mathbb{R}$ between E_1 and c' (i.e. satisfying $c' = \cos\varphi\, E_1 \circ c + \sin\varphi\, E_2 \circ c$) and compute φ'. You can use without proving it that $\varphi(L) - \varphi(0) = 2\pi$, as proven in the lecture.

c) Let ω^i be the dual 1-forms to E_i, $i = 1, 2$ (that is, $\omega^i(X) := \langle E_i, X \rangle$ for any tangent vector field X). Prove that

$$\int_{\bar{D}} \omega^1 \wedge \omega^2 = \int_{f^{-1}(\bar{D})} \sqrt{\det(g_{ij})} dx^1 dx^2 =: A(\bar{D}),$$

where $g_{ij} = \langle \partial_i f, \partial_j f \rangle$ denotes the first fundamental form and A the area measure.

d) Define the 1-forms Ω^i_j, $i, j = 1, 2$, acting on tangent vector fields X as follows $\Omega^i_j(X) := \langle D_X E_i, E_j \rangle$, where D_X denotes the covariant derivative.[1] Prove that $\Omega^i_j = -\Omega^j_i$ and deduce from b) that

$$d\Omega^1_2 = K\, \omega^1 \wedge \omega^2.$$

[1] Recall $D_X E_i(p) := \big((E_i \circ \tilde{c})'(0)\big)^T$ for any curve $\tilde{c}$ with $\tilde{c}'(0) = X_p$, where $(\,\cdot\,)^T$ is the orthogonal projection onto the tangent space TM_p

2. Isoperimetric problem on a Cartan-Hadamard surface

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface, such that f is a homeomorphism between $\mathbb{R}^2$ and $M := f(\mathbb{R}^2)$. Assume that f has nonnegative Gauss curvature K. Given $\Omega \subset M$ bounded, we say that $\partial\Omega$ is C^2 if it consists of a finite disjoint union of C^2 simple closed curves. For such Ω define the *isoperimetric quotient*

$$\mathcal{I}(\Omega) := \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)^{\frac{1}{2}}}.$$

a) Suppose first that M is isometric to the Euclidean plane. Show that if Ω_0 is a minimizer of $\mathcal{I}$ (such that $\partial\Omega_0$ is C^2) then

$$\mathcal{I}(\Omega_0) = \sqrt{4\pi} \quad \text{and } \Omega_0 \text{ is an Euclidean disc.}$$

Hint: Show that, by minimality, $\partial\Omega_0$ must consist of only one simple closed curve γ, and prove (using the first variation of arc length) that the geodesic curvature κ_g of γ must be constant.

b) For general $K \leq 0$, show that if Ω_0 is a minimizer of $\mathcal{I}$ (with $\partial\Omega_0$ of class C^2) then it must be isometric to an Euclidean disc.

Hint: Using $\Omega_r = f(B_r(0))$, with $r \to 0^+$ as competitors, show that $\mathcal{I}(\Omega_0) \leq \sqrt{4\pi}$. Show that, as in a), $\partial\Omega_0$ must consist of only one simple closed curve γ. Let ν be the inwards unit normal to $\partial\Omega_0$, define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathcal{I}(\Omega_\varepsilon) \leq 0$, and < 0 unless $K \equiv 0$ in Ω_0.

3. Weyl's tube formula

Let $U \subset \mathbb{R}^2$ be open and $f : U \to \mathbb{R}^3$ be an immersion with Gauss map $\nu : U \to \mathbb{S}^2 \subset \mathbb{R}^3$. Suppose that there is $r_\circ > 0$ such that for every point of the surface $p \in f(U)$, the points $q_+, q_- \in \mathbb{R}^3$ defined as $q_\pm := p \pm r_\circ \nu(p)$ are such that the Euclidean balls $B_{r_\circ}(q_\pm) \subset \mathbb{R}^3$ satisfy $f(U) \cap B_{r_\circ}(q_\pm) = \{p\}$ (in particular the balls are tangent to the surface at p). For $\varepsilon \in \mathbb{R}$ with $|\varepsilon| < r_\circ$ and $t \in (-\varepsilon, \varepsilon)$ define:

$$f^t(x, y) := f(x, y) + t\nu(x, y).$$

a) Show that the first fundamental form g^t_{ij} of f^t satisfies

$$\sqrt{\det(g^t_{ij}(x, y))} = \big(1 - 2tH(x, y) + t^2 K(x, y)\big)\sqrt{\det(g_{ij}(x, y))},$$

where $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K = \kappa_1\kappa_2$ are respectively the mean and Gauss curvature of f at the point (x, y) (here κ_i denote the principal curvatures), and where $g_{ij} := g^0_{ij}$ is the first fundamental form of f.

b) For f, $r_\circ$, as above show that the volume of the "cylinder" $\big\{f^t(x, y) : (x, y) \in U, t \in (-\varepsilon, \varepsilon)\big\}$, $\varepsilon \in (0, r_\circ)$ is given by

$$\iint_U \big(2\varepsilon + \tfrac{2}{3}\varepsilon^3 K(x, y)\big)\sqrt{\det(g_{ij}(x, y))}\, dx\, dy.$$

c) Prove Weyl's tube formula: let Σ be a *closed submanifold*[2] of $\mathbb{R}^3$, then for all $\varepsilon > 0$ sufficiently small, the volume of the "tube"

$$\{p \in \mathbb{R}^3 \ : \mathrm{dist}(p, \Sigma) < \varepsilon\},$$

is given by

$$2A(\Sigma)\varepsilon + \frac{4\pi}{3}\chi(\Sigma)\varepsilon^3.$$

Here $A(\Sigma)$ and $\chi(\Sigma)$ denote respectively the area and the Euler characteristic of the surface.

[2] That is, a compact submanifold without boundary, for example a sphere or a torus.

Differential Geometry I
Repetition exam (multiple choice part)

1. Assume that a (smooth, nonempty) 2-dimensional submanifold $M \subset \mathbb{R}^3$ is homeomorphic to the sphere and satisfies $\int_M H^2\, dA = 4\pi$, where H is the mean curvature[1]. Then M must be isometric to a/an:

(a) sphere.

(b) closed minimal surface.

(c) ellipsoid.

(d) Willmore's torus.

(e) small smooth perturbation of the sphere.

2. Consider the differential 2-form $\omega = -y dx \wedge dz$ in $\mathbb{R}^3$. Let D be the ellipsoid $\{(x, y, z) : (x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1\}$, where $a, b, c \in \mathbb{R} \setminus \{0\}$. Then $\int_{\partial D} \omega$ equals:

(a) $\frac{4}{3}\pi|abc|$.

(b) $\pi\sqrt{a^2 + b^2 + c^2}$.

(c) 4π.

(d) $\pi \cos a \cos b \cos c$

(e) $\pi \cosh a \cosh b \cosh c$.

[1] The average of the principal curvatures.

3. Let $C \subset \mathbb{R}^3$ be the cylinder in $\mathbb{R}^3$, parametrized as

$$f(u,v) = \big(R\cos(u+v), R\sin(u+v), v\big),$$

$R > 0$. What are the correct values of the Gauss curvature K and the mean curvature H at the point $(R, 0, \pi) \in C$ (with respect to the outward pointing Gauss map)?

(a) $K = 0, H = 0$.

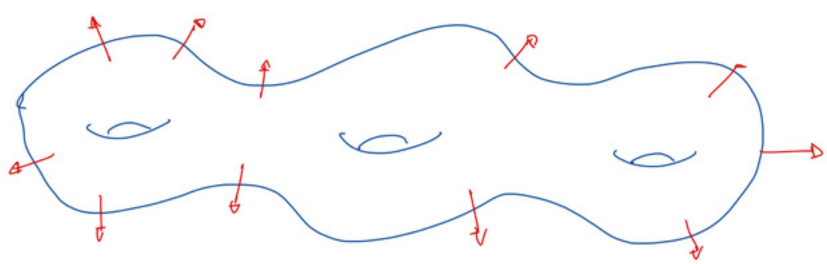
(b) $K = 0, H = -\frac{1}{2R}$.

(c) $K = 0, H = \frac{1}{2R} + 1$.

(d) $K = \frac{2}{R^2}, H = \frac{1}{R}$.

(e) $K = 0, H = -R/2$.

4. Let $M \subset \mathbb{R}^3$ be the smooth surface as depicted:



What is the value of the integral of the Gauss curvature K over M (with respect to the differential of the area)?

(a) -3π.

(b) 0.

(c) depends on how M is embedded in $\mathbb{R}^3$.

(d) -6π.

(e) -8π.

5. Consider the circle in the sphere $\mathbb{S}^2$ whose points are at (geodesic) distance $R \in (0, \pi/2)$ from the north pole. Its length and geodesic curvature (at any of its points) are, respectively

(a) $2\pi \sin R$, $\cot R$.

(b) $2\pi R$, $\frac{\cos R}{R}$.

(c) $2\pi R$, $1/R$.

(d) $2\pi \sin R$, $\tan R$.

(e) $2\pi \sin R$, $\cos R$.

6. The area of the spherical cap (containing the north pole) enclosed by the circle in the previous question is:

(a) $2\pi(1 - \cos R)$.

(b) $4\pi \sin(R/2)$.

(c) $\pi \sin^2 R$.

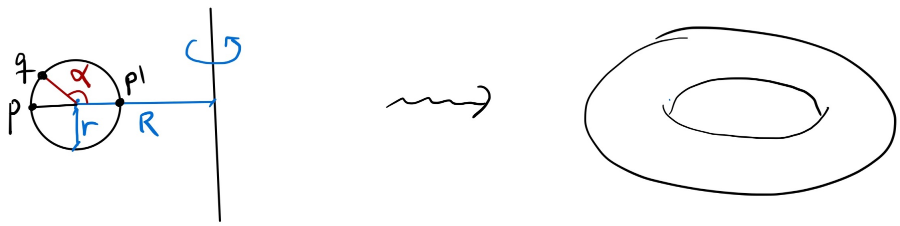
(d) $\pi \tan^2 R$.

(e) $\pi \cot R$.

7. Consider the torus of revolution

$$f(x,y) = \big(\cos x(-R + r\cos y), \sin x(-R + r\cos y), r\sin y\big),$$

$R > r > 0$, drawn below:



Its mean curvature (with respect to the outward pointing Gauss map) at $p = (-R-r, 0, 0)$ is:

(a) $-\frac{1}{2}\big(\frac{1}{r} + \frac{1}{\sqrt{R^2-r^2}}\big)$

(b) $-\frac{1}{2}\big(\frac{1}{\sqrt{rR}} + \frac{1}{R-r}\big)$.

(c) $-\frac{1}{2}\big(\frac{1}{r} + \frac{1}{R+r}\big)$.

(d) $-\frac{1}{2}\big(\frac{1}{r} - \frac{1}{R+r}\big)$.

(e) $-\frac{1}{2}\big(\frac{1}{r} - \frac{1}{\sqrt{R^2-r^2}}\big)$.

8. Consider again the torus from the previous question. At any point of the torus one principal curvature is $-1/r$. The other principal curvature at the point $q = (-R + r\cos\alpha, 0, r\sin\alpha)$ is:

(a) $\frac{\cos\alpha}{R-r\cos\alpha}$

(b) $\frac{\cos\alpha}{R-r}$

(c) $\frac{\tan\alpha}{R-r}$

(d) $\frac{\cos\alpha}{R-r\sin\alpha}$

(e) $\frac{\tan\alpha}{R+r}$

9. Consider again the torus from the previous two questions. When the point q is rotated about the x_3-axis, it generates the curve $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$, which is contained in the torus. Given a tangent vector X at q consider its parallel transport along γ for one full turn ($t \in [0, 2\pi]$), producing a new tangent vector Y at q. The angle between X and Y is:

(a) $\frac{\alpha R}{r}$

(b) $2\pi \sin\alpha$

(c) $\frac{\tan\alpha R}{r}$

(d) $2\pi \cos\alpha$

(e) $\sin\alpha$