Repetition Exam - Open Problems

1. Gauss-Bonnet and moving frames

Let $U \subset \mathbb{R}^2$ be open, $f : U \to \mathbb{R}^3$ an embedding and let $M = f(U)$.

a) Show that there exist two smooth vector fields $E_i : M \to \mathbb{R}^3$, $i = 1, 2$, which are tangent to M and satisfy $E_1 \circ f = \frac{\partial_1 f}{|\partial_1 f|}$ and $\langle E_i, E_j \rangle = \delta_{ij}$.

b) Suppose that $\bar{D} \subset M$ is homeomorphic to a disk and is bounded by a smooth unit speed curve $c : [0, L] \to M$. Let $\nu(s)$ be the unit normal to $\bar{D}$ at $c(s)$ pointing towards the interior of $\bar{D}$, and suppose that $c'(s), \nu(s)$ has the same orientation as E_1, E_2. Prove that

$$\int_D K dA = -\int_0^L \langle (E_1 \circ c)', E_2 \circ c \rangle ds,$$

where K is the Gauss curvature.

Hint: Consider a continuous angle $\varphi : [0, L] \to \mathbb{R}$ between E_1 and c' (i.e. satisfying $c' = \cos\varphi\, E_1 \circ c + \sin\varphi\, E_2 \circ c$) and compute φ'. You can use without proving it that $\varphi(L) - \varphi(0) = 2\pi$, as proven in the lecture.

c) Let ω^i be the dual 1-forms to E_i, $i = 1, 2$ (that is, $\omega^i(X) := \langle E_i, X \rangle$ for any tangent vector field X). Prove that

$$\int_{\bar{D}} \omega^1 \wedge \omega^2 = \int_{f^{-1}(\bar{D})} \sqrt{\det(g_{ij})} dx^1 dx^2 =: A(\bar{D}),$$

where $g_{ij} = \langle \partial_i f, \partial_j f \rangle$ denotes the first fundamental form and A the area measure.

d) Define the 1-forms Ω^i_j, $i, j = 1, 2$, acting on tangent vector fields X as follows $\Omega^i_j(X) := \langle D_X E_i, E_j \rangle$, where D_X denotes the covariant derivative.[1] Prove that $\Omega^i_j = -\Omega^j_i$ and deduce from b) that

$$d\Omega^1_2 = K\, \omega^1 \wedge \omega^2.$$

[1] Recall $D_X E_i(p) := \big((E_i \circ \tilde{c})'(0)\big)^T$ for any curve $\tilde{c}$ with $\tilde{c}'(0) = X_p$, where $(\,\cdot\,)^T$ is the orthogonal projection onto the tangent space TM_p

Solution. a) We can take $E_1 \circ f = \frac{\partial_1 f}{|\partial_1 f|}$, $\tilde{E}_2 \circ f = \partial_2 f - \langle E_1 \circ f, \partial_2 f \rangle (E_1 \circ f)$, and $E_2 = \frac{\tilde{E}_2}{|\tilde{E}_2|}$.

b) If φ is the angle between $E_1 \circ c$ and c' we have $c' = \cos\varphi e_1 + \sin\varphi e_2$, where $e_i := E_i \circ c$. Hence, $c'' = \cos\varphi e_1' + \sin\varphi e_2' + \varphi' \nu$, .

Using $c'' = \kappa_g \nu$ and computing the scalar product with $\nu = (-\sin\varphi e_1 + \cos\varphi e_2)$ we obtain

$$\kappa_g = \langle \cos\varphi e_1' + \sin\varphi e_2', -\sin\varphi E_1 + \cos\varphi e_2 \rangle + \varphi' = \langle e_1', e_2 \rangle + \varphi'.$$

Integrating for s between 0 and L and using Gauss-Bonnet ($\bar{D}$ is a disk) we obtain

$$2\pi - \int_{\bar{D}} K dA = \int_0^L \kappa_g \, ds = \int_0^L (\langle e_1', e_2 \rangle + \varphi') ds.$$

Using $\int_0^L \varphi' ds = \varphi(L) - \varphi(0) = 2\pi$ we conclude.

c) Let N be Gauss map along f such that $\partial_1 f, \partial_2 f, N$ is positively oriented. Writing $\partial_i f = \sum_{j=1}^2 a_i^j E_j$ we have

$$\begin{aligned}\sqrt{\det(g_{ij})} &= (\partial_1 f \times \partial_2 f) \cdot N = \big((a_1^1 E_1 + a_1^2 E_2) \times (a_2^1 E_1 + a_2^2 E_2)\big) \cdot N \\ &= (a_1^1 a_2^2 - a_1^2 a_2^1)(E_1 \times E_2) \cdot N = (\omega_1 \times \omega_2)(a_1^1 E_1 + a_1^2 E_2, a_2^1 E_2 + a_2^2 E_2) \\ &= (\omega_1 \wedge \omega_2)(\partial_1 f, \partial_2 f).\end{aligned}$$

Hence,

$$\int_{\bar{D}} \omega_1 \wedge \omega_2 = \int_{f^{-1}(\bar{D})} (\omega_1 \wedge \omega_2)(\partial_1 f, \partial_2 f) dx^1 dx^2 = \int_{f^{-1}(\bar{D})} \sqrt{\det(g_{ij})} dx^1 dx^2 = A(\bar{D}).$$

d) Note, on the one hand, that since $\Omega_2^1(X) = \langle D_X E_1, E_2 \rangle = -\langle E_1, D_X E_2 \rangle = -\Omega_2^1(X)$, proving the antisymmetry property of Ω_j^i.

Also, notice that $\Omega_2^1(c') = \langle e_1', e_2 \rangle$. Hence, follows using b) and Stokes' theorem (note that our chosen orientation of $\partial\bar{D}$ is reversed with respect to the one given by Stokes' theorem that

$$\int_{\bar{D}} K \, \omega_1 \wedge \omega_2 = \int_{\bar{D}} K dA = -\int_0^L \Omega_2^1(c') ds = \int_{\partial\bar{D}} \Omega_2^1 = \int_{\bar{D}} d\Omega_2^1.$$

Since $\bar{D}$ can be arbitrarily chosen (it can be a small neighbourhood of any point in M) we conclude $K \, \omega_1 \wedge \omega_2 = d\Omega_2^1$.

2. Isoperimetric problem on a Cartan-Hadamard surface

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface, such that f is a homeomorphism between $\mathbb{R}^2$ and $M := f(\mathbb{R}^2)$. Assume that f has nonnegative Gauss curvature K. Given $\Omega \subset M$ bounded, we say that $\partial\Omega$ is C^2 if it consists of a finite disjoint union of C^2 simple closed curves. For such Ω define the *isoperimetric quotient*

$$\mathcal{I}(\Omega) := \frac{\operatorname{length}(\partial\Omega)}{\operatorname{area}(\Omega)^{\frac{1}{2}}}.$$

a) Suppose first that M is isometric to the Euclidean plane. Show that if Ω_0 is a minimizer of $\mathcal{I}$ (such that $\partial\Omega_0$ is C^2) then

$$\mathcal{I}(\Omega_0) = \sqrt{4\pi} \quad \text{and } \Omega_0 \text{ is an Euclidean disc.}$$

Hint: Show that, by minimality, $\partial\Omega_0$ must consist of only one simple closed curve γ, and prove (using the first variation of arc length) that the geodesic curvature κ_g of γ must be constant.

b) For general $K \le 0$, show that if Ω_0 is a minimizer of $\mathcal{I}$ (with $\partial\Omega_0$ of class C^2) then it must be isometric to an Euclidean disc.

Hint: Using $\Omega_r = f(B_r(0))$, with $r \to 0^+$ as competitors, show that $\mathcal{I}(\Omega_0) \le \sqrt{4\pi}$. Show that, as in a), $\partial\Omega_0$ must consist of only one simple closed curve γ. Let ν be the inwards unit normal to $\partial\Omega_0$, define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \operatorname{image}(\gamma_\varepsilon)$. Show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathcal{I}(\Omega_\varepsilon) \le 0$, and < 0 unless $K \equiv 0$ in Ω_0.

Solution. a) We can assume without loss of generality $M = \mathbb{R}^2$, since the isoperimetric problem is intrinsic. Notice first that any minimizer of $\mathcal{I}$ must be connected, since the numerator is additive, $\operatorname{length}(\partial \cup \Omega_i) = \sum_i \operatorname{length}(\partial\Omega_i)$, and the denominator subadditive $\operatorname{area}(\cup_i\Omega_i)^{\frac{1}{2}} \le \sum_i \operatorname{area}(\Omega_i)^{\frac{1}{2}}$, with equality if and only if the number of components is one.

Note also that if $\partial\Omega_0$ has multiple components each is a closed simple curve. Hence, the image of each of these curves divides $\mathbb{R}^2$ into two connected components (one bounded and one unbounded). Now, the union of (the closures of) the bounded components is a new set which contains Ω_0 and whose boundary is contained in $\partial\Omega_0$. Hence, this set obtained by "filling the holes" would have more area and less perimeter, contradicting the fact that Ω_0 minimizes $\mathcal{I}$.

Let $\gamma : [0, L] \to \mathbb{R}^2$ be a curve tracing $\partial\Omega_0$, parametrized by the arc length, and let $\nu : [0, L] \to \mathbb{S}^1$ be the inwards unit normal. Given $\xi \in C^2_{\text{closed}}([0, L])$

define $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\xi(t)\nu(t)$ and let Ω_ε be the bounded connected component of $\mathbb{R}^2 \setminus \text{image}(\gamma_\varepsilon)$. If $\int_0^L \xi(t)dt = 0$ then $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{area}(\Omega_\varepsilon) = 0$. Hence be minimality it must be $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{length}(\Omega_\varepsilon) = -\int_0^L \kappa_g(t)\xi(t)dt = 0$. Since ξ is an arbitrary average zero smooth function we deduce that $\kappa_g \equiv \kappa = \textit{constant}$ or equivalently $\gamma'' \equiv \kappa\nu$. This easily implies that γ traces a circle with radius $1/\kappa$.

b) Reparametrize $f(\mathbb{R}^2)$ as $\tilde{f}(x) = f(Ax)$, where A is a matrix with nonzero determinant, so that $g_0 = \text{Id}$, so $|g_x - \text{Id}| \le Cr$ for $|x| < r$. Then, $\text{area}(\tilde{f}(B_r(0))) = \pi r^2(1 + o(1))$ and $\text{length}(\tilde{f}(\partial B_r(0))) = 2\pi r(1 + o(1))$. It follows that

$$\mathcal{I}(\tilde{f}(B_r(0)) = \sqrt{4\pi} + o(1) \quad \text{as } r \downarrow 0.$$

Since by assumption Ω_0 is a minimizer of $\mathcal{I}$ it must be $\mathcal{I}(\Omega_0) \le \mathcal{I}(\tilde{f}(B_r(0))$ for all $r > 0$ and hence $\mathcal{I}(\Omega_0) \le \sqrt{4\pi}$.

As in a) —now using that M is homeomophic to $\mathbb{R}^2$—, $\partial\Omega_0$ must consist of only one simple closed curve $f\circ\gamma$. Let us take γ oriented counterclockwise and let $\nu(t)$ be the unit normal to $\partial\Omega_0$ at $(f \circ \gamma)(t)$ pointing towards the interior (as in the Gauss-Bonnet setting). Define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Let us show that show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathcal{I}(\Omega_\varepsilon) \le 0$, and < 0 unless $K \equiv 0$ in Ω_0.

Indeed, on the one hand $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{area}(\Omega_\varepsilon) = -\text{length}(\partial\Omega_0)$. On the other hand, $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{length}(\partial\Omega_\varepsilon) = -\int_{\partial\Omega_0} \kappa_g ds$

Now, using Gauss-Bonnet, $\int_{\partial\Omega_0} \kappa_g ds = 2\pi - \int_{\Omega_0} KdA \ge 2\pi$ ($>2\pi$ unless $K \equiv 0$). Hence,

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{I}(\Omega_\varepsilon) &= \frac{\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{length}(\partial\Omega_\varepsilon)}{\text{area}(\Omega_0)^{\frac{1}{2}}} - \frac{1}{2}\frac{\text{length}(\partial\Omega_0)\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{area}(\Omega_\varepsilon)}{\text{area}(\Omega_0)^{\frac{3}{2}}} \\
&\le (<) -\frac{2\pi}{\text{area}(\Omega_0)^{\frac{1}{2}}} + \frac{1}{2}\frac{\text{length}(\partial\Omega_0)^2}{\text{area}(\Omega_0)^{\frac{3}{2}}} \\
&= -\frac{2\pi}{\text{area}(\Omega_0)^{\frac{1}{2}}} + \frac{1}{2}\frac{\mathcal{I}(\Omega_0)^2}{\text{area}(\Omega_0)^{\frac{1}{2}}} \le 0,
\end{aligned}$$

since $\mathcal{I}(\Omega_0)^2 \le 4\pi$. This contradicts the minimality of Ω_0 unless the second inequality is an equality, which implies that $K \equiv 0$ in Ω_0.

3. Weyl's tube formula

Let $U \subset \mathbb{R}^2$ be open and $f : U \to \mathbb{R}^3$ be an immersion with Gauss map $\nu : U \to \mathbb{S}^2 \subset \mathbb{R}^3$. Suppose that there is $r_\circ > 0$ such that for every point of the surface $p \in f(U)$, the points $q_+, q_- \in \mathbb{R}^3$ defined as $q_\pm := p \pm r_\circ \nu(p)$ are such that the Euclidean balls $B_{r_\circ}(q_\pm) \subset \mathbb{R}^3$ satisfy $f(U) \cap B_{r_\circ}(q_\pm) = \{p\}$ (in particular the balls are tangent to the surface at p). For $\varepsilon \in \mathbb{R}$ with $|\varepsilon| < r_\circ$ and $t \in (-\varepsilon, \varepsilon)$ define:

$$f^t(x, y) := f(x, y) + t\nu(x, y).$$

a) Show that the first fundamental form g^t_{ij} of f^t satisfies

$$\sqrt{\det(g^t_{ij}(x, y))} = \big(1 - 2tH(x, y) + t^2 K(x, y)\big)\sqrt{\det(g_{ij}(x, y))},$$

where $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K = \kappa_1\kappa_2$ are respectively the mean and Gauss curvature of f at the point (x, y) (here κ_i denote the principal curvatures), and where $g_{ij} := g^0_{ij}$ is the first fundamental form of f.

b) For f, $r_\circ$, as above show that the volume of the "cylinder" $\big\{f^t(x, y) : (x, y) \in U, t \in (-\varepsilon, \varepsilon)\big\}$, $\varepsilon \in (0, r_\circ)$ is given by

$$\iint_U \big(2\varepsilon + \tfrac{2}{3}\varepsilon^3 K(x, y)\big)\sqrt{\det(g_{ij}(x, y))}\, dx\, dy.$$

c) Prove Weyl's tube formula: let Σ be a *closed submanifold*[2] of $\mathbb{R}^3$, then for all $\varepsilon > 0$ sufficiently small, the volume of the "tube"

$$\{p \in \mathbb{R}^3 \ : \operatorname{dist}(p, \Sigma) < \varepsilon\},$$

is given by

$$2A(\Sigma)\varepsilon + \frac{4\pi}{3}\chi(\Sigma)\varepsilon^3.$$

Here $A(\Sigma)$ and $\chi(\Sigma)$ denote respectively the area and the Euler characteristic of the surface.

[2] That is, a compact submanifold without boundary, for example a sphere or a torus.

Solution: a) Given $z = (x, y) \in U$ choose and orthonormal basis e_1, e_2 of (TU_z, g_z) consisting of eigenvectors of the Weingarten map L_z. In the new coordinates ($\tilde{f}(x^1, x^2) =: f(z+x^1e_1+x^2e_2)$) we have $\partial_i\tilde{\nu}(0) = \sum_{k=1}^2 \tilde{h}^k{}_i \cdot \partial_k \tilde{f} = -\kappa_i\partial_i f$ and hence $\partial_i\tilde{f}^t_i = \partial_i\tilde{f}_i(1 - t\kappa_i)$. Therefore

$$\begin{aligned}\tilde{g}^t_{ij}(0) &= \langle \partial_i\tilde{f}^t(0), \partial_j\tilde{f}^t(0)\rangle = (1 - t\kappa_i)(1 - t\kappa_j)\partial_i\tilde{f}(0), \partial_j\tilde{f}(0)\rangle \\ &= (1 - t\kappa_i)(1 - t\kappa_j)\tilde{g}_{ij}(0).\end{aligned}$$

Hence, using $\tilde{g}_{ij}(0) = \delta_{ij}$ we obtain $\det(\tilde{g}^t_{ij}(0)) = (1 - \kappa_1)^2(1 - t\kappa_2)^2$ and so, after performing the change of basis

$$\det(g^t(z)) = (1 - \kappa_1)^2(1 - t\kappa_2)^2\det(g(z)).$$

Taking the square root we conclude.

b) Consider the injective smooth parametrization of the cylinder

$$F(x, y, t) := f(x, y) + t\nu(x, y)$$

with $(x, y, t) \in U \times (-\varepsilon, \varepsilon)$.

Observe that

$$|\det(DF(x, y, t))| = |(\partial_x F \times \partial_y F) \cdot \partial_t F| = |(\partial_x f^t \times \partial_y f^t) \cdot \nu| = \sqrt{\det(g^t(z))}$$

Therefore, using a) the volume of the cylinder is given by

$$\iint_U \int_{-\varepsilon}^{\varepsilon} \big(1 - 2tH(x, y) + t^2K(x, y)\big)\sqrt{\det(g_{ij}(x, y))}\, dtdx\ dy.$$

Integrating first with respect to the variable t (for fixed $(x, y) \in U$) we obtain the desired formula.

c) Since Σ is a closed (compact) orientable surface it admits some finite Atlas $\{(V_\alpha, \psi_\alpha)\}$, where $V_\alpha \subset \Sigma$ are open sets and $\psi_\alpha : V_\alpha \to U_\alpha \subset \mathbb{R}^2$. Also, by compactness, there is $r_\circ > 0$ such that every point $p \in \Sigma$ admits tangent Euclidean balls of radius $r_\circ$ whose intersection with Σ is $\{p\}$ from both sides Notice that $f_\alpha := \psi_\alpha^{-1}$ are embeddings of finitely many open pieces of Σ which cover all of Σ.

For $V \subset \Sigma$ and $\varepsilon \in (0, r_\circ)$, let $\mathcal{C}_\varepsilon(V)$ denote the "cylinder above V" defined as $\mathcal{C}_\varepsilon(V) := \big\{p + t\nu(p)\ :\ p \in V, t \in (-\varepsilon, \varepsilon)\big\}$,

By *b*), if V is any open subset of some of the V_α we have

$$|\mathcal{C}_\varepsilon(V)| = \int_V (2\varepsilon + \tfrac{2}{3}K\varepsilon^3)dA,$$

where we denote by $|E|$ the volume of a measurable subset $E \subset \mathbb{R}^3$.

Now, thanks to the inclusion exclusion principle

$$\left|\bigcup_{\alpha=1}^{N} E_i\right| = \sum_{\alpha=1}^{N} |E_\alpha| - \sum_{1\leq\alpha<\beta\leqslant N} |E_\alpha \cap E_\beta| + \sum_{1\leqslant\alpha<\beta<\gamma\leqslant N} |E_\alpha \cap E_\beta \cap E_\gamma| \\ - \quad \cdots \quad + (-1)^{N+1} |E_1 \cap \cdots \cap E_N|,$$

we can apply the formula from b) to each $\mathcal{C}_\varepsilon(V_\alpha)$, to the intersections $\mathcal{C}_\varepsilon(V_\alpha \cap V_\beta) = \mathcal{C}_\varepsilon(V_\alpha) \cap \mathcal{C}_\varepsilon(V_\beta)$, and all other possible intersection as well, to obtain

$$|\mathcal{C}_\varepsilon(\Sigma)| = \int_\Sigma (2\varepsilon + \tfrac{2}{3}K\varepsilon^3)d\sigma = 2\varepsilon A(\Sigma) + \tfrac{2}{3}\varepsilon^3 \int_\Sigma K \, dA$$

Since by Gauss-Bonnet we have $\int_\Sigma K dA = 2\pi\chi(\Sigma)$ we conclude.

Differential Geometry I
Repetition exam (multiple choice part)

1. Assume that a (smooth, nonempty) 2-dimensional submanifold $M \subset \mathbb{R}^3$ is homeomorphic to the sphere and satisfies $\int_M H^2\, dA = 4\pi$, where H is the mean curvature[1]. Then M must be isometric to a/an:

√ (a) sphere.

(b) closed minimal surface.

(c) ellipsoid.

(d) Willmore's torus.

(e) small smooth perturbation of the sphere.

Solution. If k_1, k_2 denote the principal curvatures of M, then $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$. We compute

$$4H^2 = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1k_2 = (k_1 - k_2)^2 + 4K$$

Hence $4\pi = \int_M H^2 \geq \int_M K = 2\pi\chi(M) = 4\pi(1-g)$, where g is the genus of the connected closed surface M. Since $g = 0$ (M homeomorphic to a sphere) we must have $k_1 = k_2$ at every point. That is every point must be umbilical. Hence (see Theorem 4.6 in the lecture) the surface must be isometric to a sphere.

2. Consider the differential 2-form $\omega = -y dx \wedge dz$ in $\mathbb{R}^3$. Let D be the ellipsoid $\{(x, y, z) : (x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1\}$, where $a, b, c \in \mathbb{R} \setminus \{0\}$. Then $\int_{\partial D} \omega$ equals:

√ (a) $\frac{4}{3}\pi|abc|$.

(b) $\pi\sqrt{a^2 + b^2 + c^2}$.

(c) 4π.

(d) $\pi \cos a \cos b \cos c$

(e) $\pi \cosh a \cosh b \cosh c$.

Solution. By Stokes' theorem $\int_{\partial D} \omega = \int_D d\omega = \int_D dx \wedge dy \wedge dz$, that is the volume of D. Since the three semiaxis of the ellipsoid are $|a|, |b|, |c|$ its volume is $\frac{4}{3}\pi|abc|$.

[1] The average of the principal curvatures.

3. Let $C \subset \mathbb{R}^3$ be the cylinder in $\mathbb{R}^3$, parametrized as

$$f(u, v) = \big(R\cos(u + v), R\sin(u + v), v\big),$$

$R > 0$. What are the correct values of the Gauss curvature K and the mean curvature H at the point $(R, 0, \pi) \in C$ (with respect to the outward pointing Gauss map)?

(a) $K = 0, H = 0$.

√ (b) $K = 0, H = -\frac{1}{2R}$.

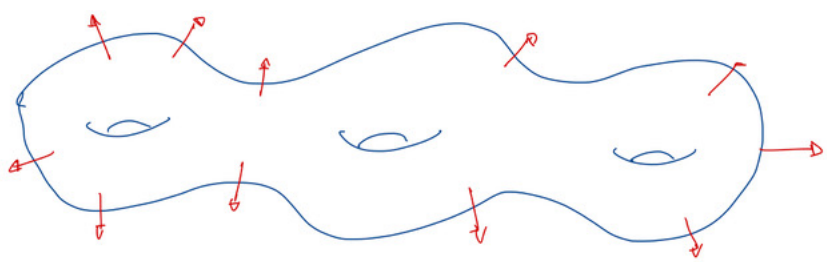
(c) $K = 0, H = \frac{1}{2R} + 1$.

(d) $K = \frac{2}{R^2}, H = \frac{1}{R}$.

(e) $K = 0, H = -R/2$.

Solution Being a cylinder of radius R, at any point the principal curvatures are $k_1 = -1/R$ and $k_2 = 0$. Thus $K = 0$ and $H = -\frac{1}{2R}$.

4. Let $M \subset \mathbb{R}^3$ be the smooth surface as depicted:



What is the value of the integral of the Gauss curvature K over M (with respect to the differential of the area)?

(a) -3π.

(b) 0.

(c) depends on how M is embedded in $\mathbb{R}^3$.

(d) -6π.

√ (e) -8π.

Solution The sketch shows a genus three surface, so its Euler characteristic is 2(1-3) = -4. Hence, by Gauss-Bonnet $\int_M K dA = 2\pi\chi(M) = -8\pi$.

5. Consider the circle in the sphere $\mathbb{S}^2$ whose points are at (geodesic) distance $R \in (0, \pi/2)$ from the north pole. Its length and geodesic curvature (at any of its points) are, respectively

√ (a) $2\pi \sin R$, $\cot R$.

(b) $2\pi R$, $\frac{\cos R}{R}$.

(c) $2\pi R$, $1/R$.

(d) $2\pi \sin R$, $\tan R$.

(e) $2\pi \sin R$, $\cos R$.

Solution The radius of the planar disc spanned by the circle is $\sin R$, so its length $2\pi \sin R$. Its total curvature is hence $1/\sin R$ but since the angle between the acceleration vector and the tangent plane is R the geodesic curvature is $\cos R/\sin R$.

6. The area of the spherical cap (containing the north pole) enclosed by the circle in the previous question is:

√ (a) $2\pi(1 - \cos R)$.

(b) $4\pi \sin(R/2)$.

(c) $\pi \sin^2 R$.

(d) $\pi \tan^2 R$.

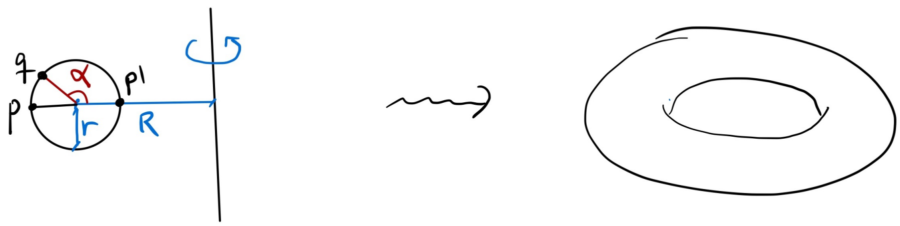
(e) $\pi \cot R$.

Solution Calling C the spherical cap we have, by Gauss-Bonnet $A(C)+2\pi \sin R \cot R = \int_C K dA + L(\partial C)\kappa_g = 2\pi\chi(C) = 2\pi$.

7. Consider the torus of revolution

$$f(x,y) = \big(\cos x(-R + r\cos y), \sin x(-R + r\cos y), r\sin y\big),$$

$R > r > 0$, drawn below:



Its mean curvature (with respect to the outward pointing Gauss map) at $p = (-R-r, 0, 0)$ is:

(a) $-\frac{1}{2}\left(\frac{1}{r} + \frac{1}{\sqrt{R^2-r^2}}\right)$

(b) $-\frac{1}{2}\left(\frac{1}{\sqrt{rR}} + \frac{1}{R-r}\right)$.

✓ (c) $-\frac{1}{2}\left(\frac{1}{r} + \frac{1}{R+r}\right)$.

(d) $-\frac{1}{2}\left(\frac{1}{r} - \frac{1}{R+r}\right)$.

(e) $-\frac{1}{2}\left(\frac{1}{r} - \frac{1}{\sqrt{R^2-r^2}}\right)$.

Solution At the point p both principal curvatures have the same sign. One corresponds to the meridian and equals $-\frac{1}{r}$ (the meridian has radius r). The other corresponds to the parallel through p, which is a geodesic (by symmetry) traces a circumference of radius $R+r$, so the curvature gives $-\frac{1}{R+r}$. Then mean curvature is its average of the two principal curvatures computed before.

8. Consider again the torus from the previous question. At any point of the torus one principal curvature is $-1/r$. The other principal curvature at the point $q = (-R + r\cos\alpha, 0, r\sin\alpha)$ is:

√ (a) $\frac{\cos\alpha}{R - r\cos\alpha}$

(b) $\frac{\cos\alpha}{R-r}$

(c) $\frac{\tan\alpha}{R-r}$

(d) $\frac{\cos\alpha}{R - r\sin\alpha}$

(e) $\frac{\tan\alpha}{R+r}$

Solution The principal curvature $-\frac{1}{r}$ corresponds to the meridians. To compute the principal curvature in the orthogonal direction at q we consider the circumference $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$. Its curvature is $1/(R - r\cos\alpha)$, so after projecting (the normal vector to the circumference at q points in the (1,0) while the normal to the surface is $(\cos\alpha, \sin\alpha)$ at q) we obtain that the normal curvature is $\frac{\cos\alpha}{R - r\cos\alpha}$ direction. Since γ is a curve trough q contained in the surface and velocity vector orthogonal to the other principal direction, its normal curvature is precisely the principal curvature we are trying to compute.

9. Consider again the torus from the previous two questions. When the point q is rotated about the x_3-axis, it generates the curve $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$, which is contained in the torus. Given a tangent vector X at q consider its parallel transport along γ for one full turn ($t \in [0, 2\pi]$), producing a new tangent vector Y at q. The angle between X and Y is:

(a) $\frac{\alpha R}{r}$

√ (b) $2\pi\sin\alpha$

(c) $\frac{\tan\alpha R}{r}$

(d) $2\pi\cos\alpha$

(e) $\sin\alpha$

Solution Consider the cone tangent to the torus along γ. It is a cone of revolution (also with respect the x_3 axis) and the angle of its generating lines of the cone and the x_3 axis is $\sin\alpha$. Hence when "opening" the cone (as we saw in the lecture in the it becomes a flat example of Foucault's pendulum) it becomes a flat circular sector of angle $2\pi\sin\alpha$. Hence, since parallel transport is trivial for the flat surface, we see that the angle between the transported vector and the original one is $2\pi\sin\alpha$.