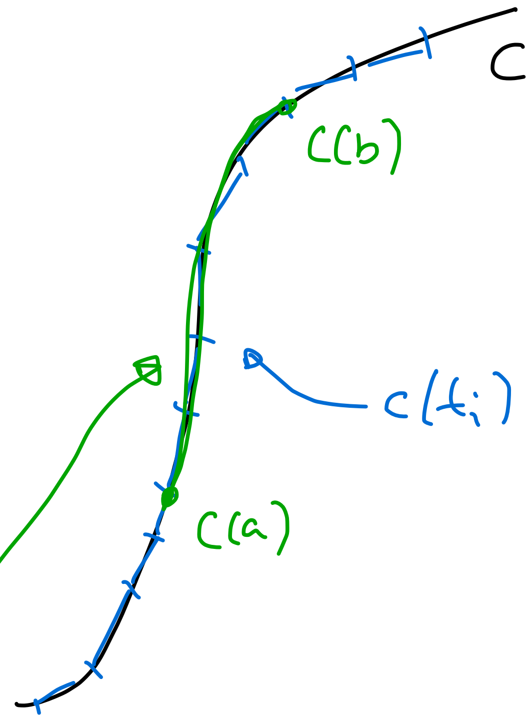
Ch 1 Curves $\mathbb{R}^3$, $|\cdot|$

$$|x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

$$d(x,y) = |x-y|$$

$$c : I \to \mathbb{R}^3$$

interval $\subset \mathbb{R}$



$$L(c) = \sup \sum_{i=0}^{k-1} |c(t_{i+1}) - c(t_i)|$$

among $t_0 \le t_1 \le t_2 \le \dots \le t_k$, $t_i \in I$

$[a,b] \subset I$

$$L(c|_{[a,b]})$$

exercise

$a < r < b$

$$\boxed{L(c|_{[a,b]}) = L(c|_{[a,r]}) + L(c|_{[r,b]})}$$

def'n constant speed param $c: I \longrightarrow \mathbb{R}^3$

$$L(c|_{[a,b]}) = \lambda(b-a) \qquad (*)$$

$\tilde{c}: \tilde{I} \to \mathbb{R}^3$ is a re-parametrization of c if $\exists \varphi: I \to \tilde{I}$

$c = \tilde{c} \circ \varphi$ $\qquad \varphi$ is bijetive & cont. $a<b \Rightarrow \varphi(a) \leq \varphi(b)$

$<$

Lemma 1.1 (reparam.) For any $c: I \longrightarrow \mathbb{R}^3$ with $L(c|_{[ab]}) < \infty$, $\forall [a,b] \subset I$, $\exists \tilde{c}$ reparam. with

unit speed ($\Longleftrightarrow$ $(*)$ with $\lambda = 1$)

Fix $t_0 \in I$ "choose origin of new time"

$$\varphi(t) = \begin{cases} L(c|_{[t_0,t]}) & t \geq t_0 \\ -L(c|_{[t,t_0]}) & t \leq t_0 \end{cases}$$

We will check φ is bijective & continuous and then define $\boxed{\tilde{c} \circ \varphi = c}$ $\boxed{\tilde{c} = c \circ \varphi^{-1}}$

φ is nondecreasing

$\Rightarrow$ $\varphi(b) - \varphi(a) = L(c|_{[a,b]}) \geq |c(b) - c(a)|$

> 0

unless $c(b) = c(a)$

φ is <u>continuous</u> (let me show right-cont)

$$|\varphi(a+r) - \varphi(a)| < \delta \quad \text{if} \quad r > 0 \quad \text{sufficiently small}$$

$$0 \le \varphi(a+r) - \varphi(a) = L(c|_{[a,a+r]})$$

$$= L(c|_{[a,b]}) - \overbrace{L(c|_{[a+r,b]})}$$

$$\le \sum_{i=0}^{k-1} |c(t_{i+1}) - c(t_i)| + \frac{\delta}{2} - L(c|_{[a+r,b]})$$

use def'n $L(c|_{[a,b]}) < \infty$

I will choose $r \in [t_0, t_1)$

$$\le |c(t_0) - c(r)| + \overbrace{|c(r) - c(t_1)|} + \underbrace{\sum_{i=1}^{k-1} |c(t_{i+1}) - c(t_i)|} + \frac{\delta}{2} - L(c|_{[a+r,b]})$$

def'n of $L(c|_{[r,b]})$ $\leq |c(t_0) - c(r)| + \frac{\delta}{2}$

c cont. $\leq \frac{\delta}{2} + \frac{\delta}{2}$

When $c : I \to \mathbb{R}^3$ is of class C^1

c is <u>regular</u> if $c' \neq 0$

If c is regular then c has constant speed $\Leftrightarrow$

$$|c'(t)| = \lambda > 0$$

check this

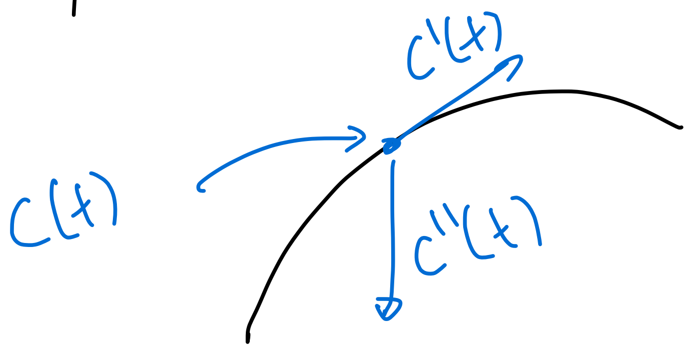
Exercises

- If c is C^1 $\quad L(c|_{[a,b]}) = \int_a^b |c'(t)|\, dt$

Local theory of curves $\mathbb{R}^2, \mathbb{R}^3$ $\quad (\mathbb{R}^n)$

Frenet curves (Def'n 1.2) $\quad c$ any C^2 curve

- In $\mathbb{R}^2$ any regular curve
- In $\mathbb{R}^3$ any regular curve with $c'(t)$, $c''(t)$ l.i for all t



Frenet frame

$\mathbb{R}^2$ $\quad e_1(t) \quad e_2(t)$ orthon. posit. oriented

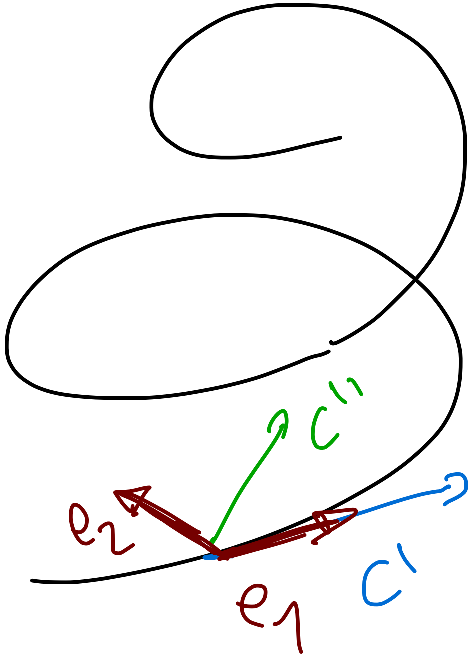
$e_1 \parallel c'$ $\qquad e_1 \cdot c' = |c'| > 0$

thus $e_1 = \dfrac{c'(t)}{|c'(t)|}$

$\mathbb{R}^3$ $\quad e_1(t), e_2(t), e_3(t)$ orth. posit. oriented basis of $\mathbb{R}^3$

$e_1 = \dfrac{c'(t)}{|c'(t)|}$, $\quad \mathrm{span}(e_1, e_2) = \mathrm{span}(c', c'')$

$e_2 \cdot c'' > 0$



$e_3 := e_1 \times e_2$

Frenet curvatures $(\mathbb{R}^3)$

$$(**) \quad \kappa(t) = \frac{1}{|c'(t)|}\left(e_1'(t) \cdot e_2(t)\right) > 0$$

$$\tau(t) = \frac{1}{|c'(t)|}\left(e_2'(t) \cdot e_3(t)\right)$$

In $\mathbb{R}^2$ $\quad K_{or} = \frac{1}{|c'(t)|}\left(e_1'(t)\cdot e_2(t)\right) \begin{cases} >0 \\ <0 \\ =0 \end{cases}$

Prop. 1.4

$$\frac{1}{|c'(t)|}\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \quad \text{(FE)}$$

Compute

$$\frac{1}{|c'|} e_1' = \kappa e_2$$

$$\frac{1}{|c'|} e_1'\cdot e_2 = \kappa \quad \Longleftrightarrow \quad (**)$$

$$e_1\cdot e_1 = 1 \Longrightarrow e_1'\cdot e_1 + e_1\cdot e_1' = 0$$

$$\Longrightarrow e_1'\cdot e_1 = 0$$

To see that the metrix is antisymmetric

$$\frac{d}{dt} \quad e_i(t) \cdot e_j(t) = \delta_{ij}$$

$$e_i' \cdot e_j + e_i \cdot e_j' = 0$$

I want to prone

$$\frac{1}{c'(t)} e_2' = -\kappa e_1 + \tau e_3$$

$$\frac{1}{c'(t)} e_2' \cdot e_3 =: \tau(t)$$

Thm 1.5 (Fund. thm of local curve theory) $\mathbb{R}^3$

Given $\kappa : I \to (0,\infty)$, $\tau : I \to \mathbb{R}$ of class C^∞, $s_0 \in I$, $x_0 \in \mathbb{R}^3$, (b_1, b_2, b_3) pos. orthonormal basis of $\mathbb{R}^3$

$\exists$ unique Frenet curve $c \in C^\infty(I, \mathbb{R}^3)$ of speed one s.t

(1) $c(s_0) = x_0$

(2) (b_1, b_2, b_3) is Frenet frame of c at x_0

(3) κ, τ are Frenet curv and torsion of c

1. Construct $(e_1(s), e_2(s), e_3(s))$ solving (FE)

$$(e_1(s_0), e_2(s_0), e_3(s_0)) = (b_1, b_2, b_3)$$

(unique sol'n b ODE theory)

$E' = KE \qquad E = E(s),\ K = K(s) \qquad E(s_0) \in SO(3) \qquad K(s) + K^T(s) = 0$

$\Longrightarrow E(s) \in SO(3) \ \forall s$

$$(EE^T)' = E'E^T + E(E')^T = K(EE^T) + (EE^T)K^T \qquad (EE^T)(s_0) = id$$

$c'(t) = e_1(t)$

2. If c will be unit speed

So let us define $\qquad c(t) := \int_{s_0}^{t} e_1(s)\,ds + x_0$

3. Check c satisfies all promised properties

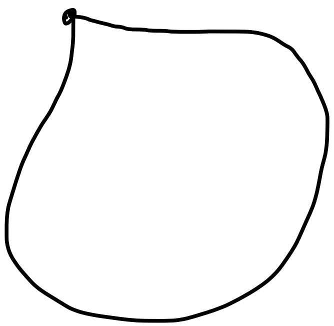
$$c' = e_1 \qquad c'' = e_1' = \kappa e_2$$

The rotation index of a plane curve

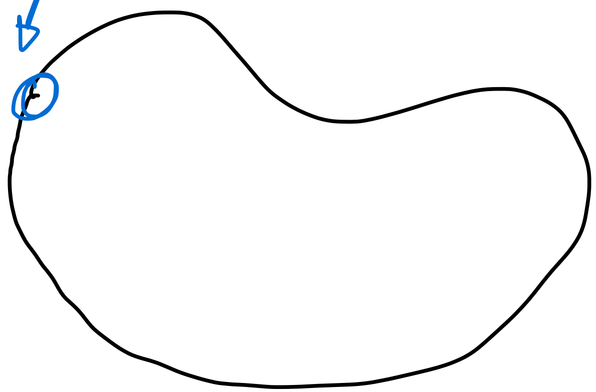
$c : [a,b] \longrightarrow \mathbb{R}^n$ is a closed C^k curve

If $\exists\, \tilde{c} : \mathbb{R} \longrightarrow \mathbb{R}^n$ $\tilde{c}$ is C^k and $(b-a)$-periodic with $\tilde{c}|_{[a,b]} = c$

C^0- Closed



all derivatives up to k-th agree

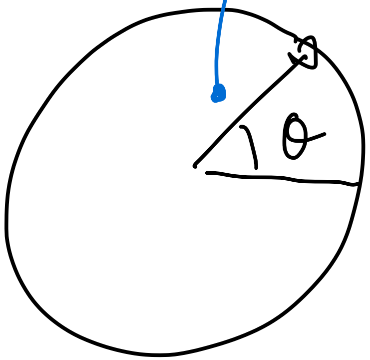


Suppose $c : [a,b] \to \mathbb{R}^2$ is <u>regular</u> and C^1-closed

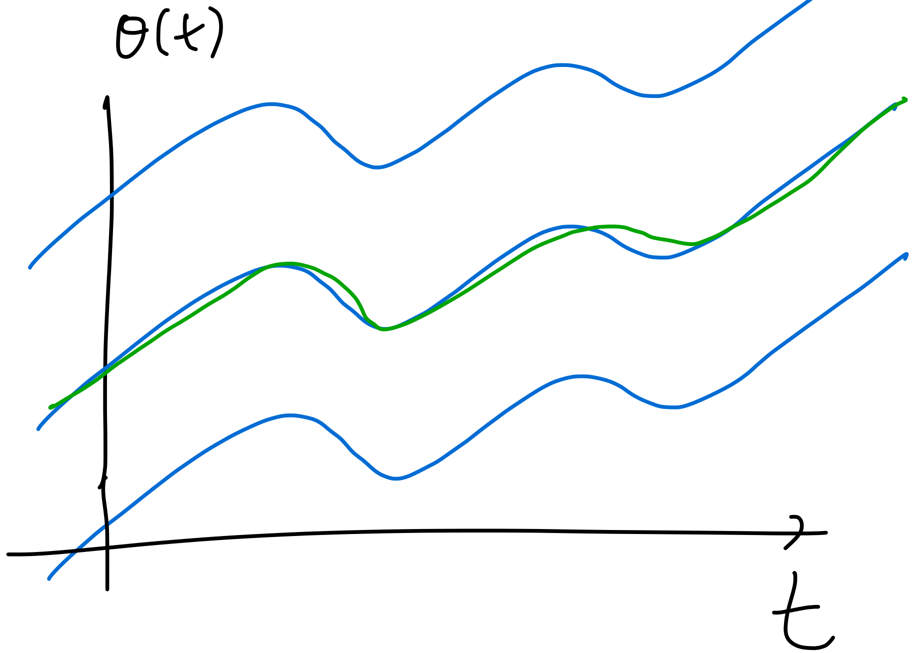
then

$\in S^1$

$$\frac{c'(t)}{|c'(t)|} = (\cos \theta(t) , \sin \theta(t))$$



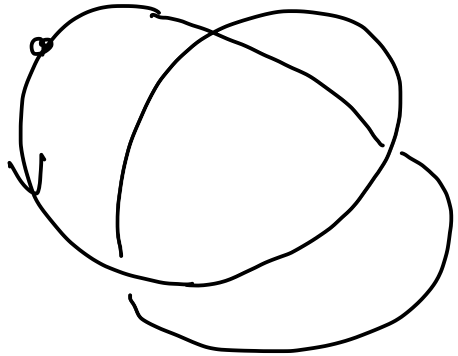
$\theta + 2\pi \mathbb{Z}$



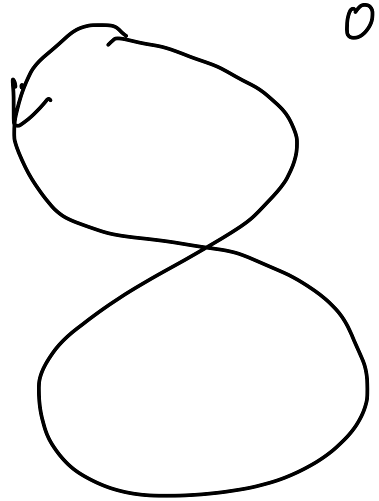
Given c as above, define

<u>rotation index</u> (<u>Umlaufzahl</u>) of c as $\dfrac{\theta(b) - \theta(a)}{2\pi}$

+2



0



−1

Remark if $c : [a,b] \to \mathbb{R}^2$ is C^2 closed unit speed

e_1, e_2 Frenet frame

$$e_1(s) = (\cos\theta(s), \sin\theta(s))$$

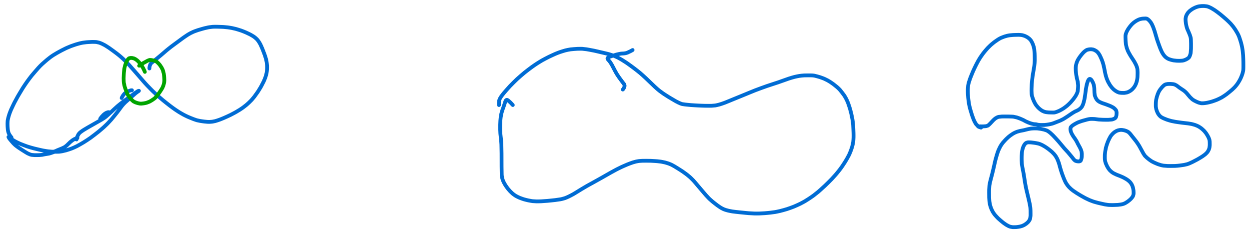
$$\Rightarrow e_1'(s) = (-\sin\theta(s), \cos\theta(s))\,\theta'(s) = e_2(s)\,\theta'(s)$$

comput sclar prod with $e_2(s)$

rotation index

$$\kappa_{or}(s) = e_1' e_2(s) = \theta'(s) \Rightarrow \theta(b) - \theta(a) = \int_a^b \theta'(s)\,ds$$

$$= \int_a^b \kappa_{or}(s)\,ds$$

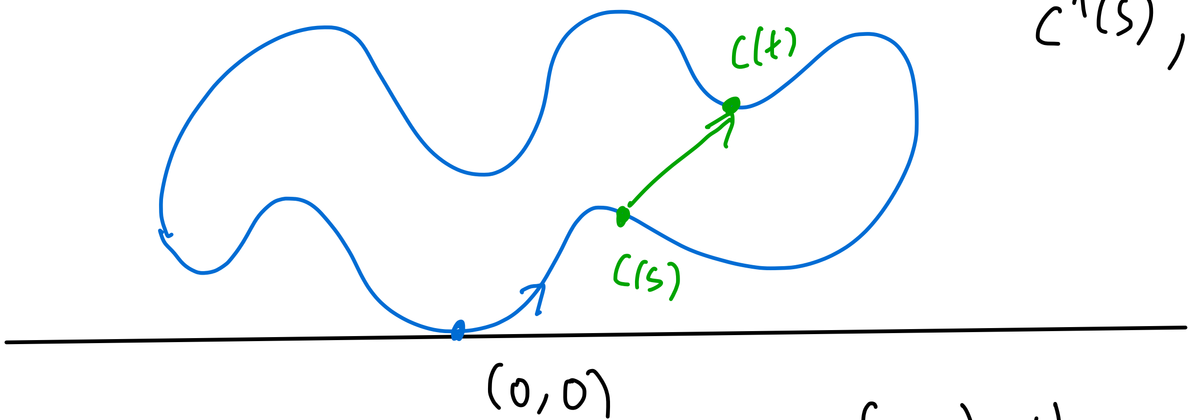


$c|_{[a,b)}$ injective

1.6 Thm ("thm of turning tangents", "Umlaufsatz")

The rotation number of any regular, simply closed C^1 curve $c : [a,b] \to \mathbb{R}^2$ is $+1$ or -1

proof (Heinz Hopf 1935) Suppose (up to reparam) $|c'| = 1$, assume $c([a,b]) \subset \mathbb{R} \times [0,\infty)$, $c(a) = (0,0)$, $c'(a) = (1,0)$



$c^1(s), c^2(s)$

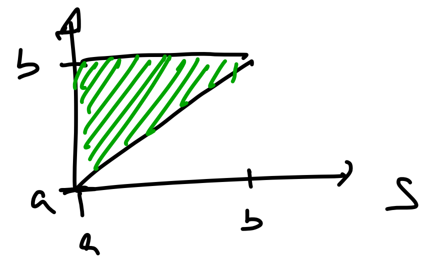
$\min_{[a,b]} c^2(s)$

Goal show rotation number $= +1$

Idea: construct clever continuous extension of $\frac{c'(t)}{|c'(t)|}$

$$e(s,t) = \begin{cases} \dfrac{c(t)-c(s)}{|c(t)-c(s)|} & \text{if } a \le s < t < b \\[2ex] \dfrac{c'(t)}{|c'(t)|} & \text{if } a \le s = t < b \\[2ex] (-1,0) & \text{if } s=a,\ t=b \end{cases}$$

$$\boxed{D} = \{ (s,t) \in \mathbb{R}^2 \mid a \le s \le t \le b \}$$



$e : D \longrightarrow S^1$ continuous

$e(s,t) = (\cos\theta(s,t), \sin\theta(s,t))$

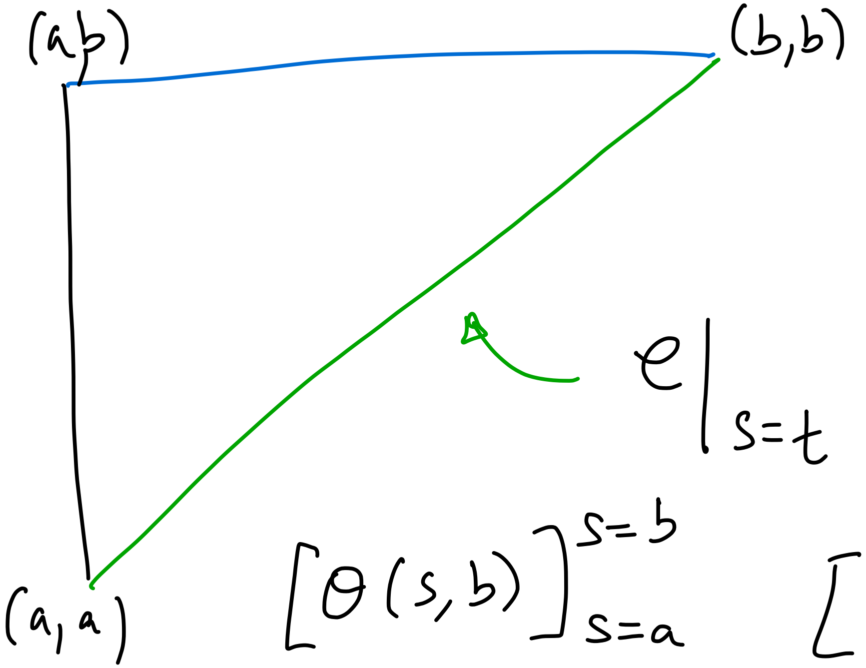
$\theta : D \longrightarrow \mathbb{R}$

<u>is continuous</u>

rotation number

$$\| $$

$$\frac{\theta(b,b) - \theta(a,a)}{2\pi}$$



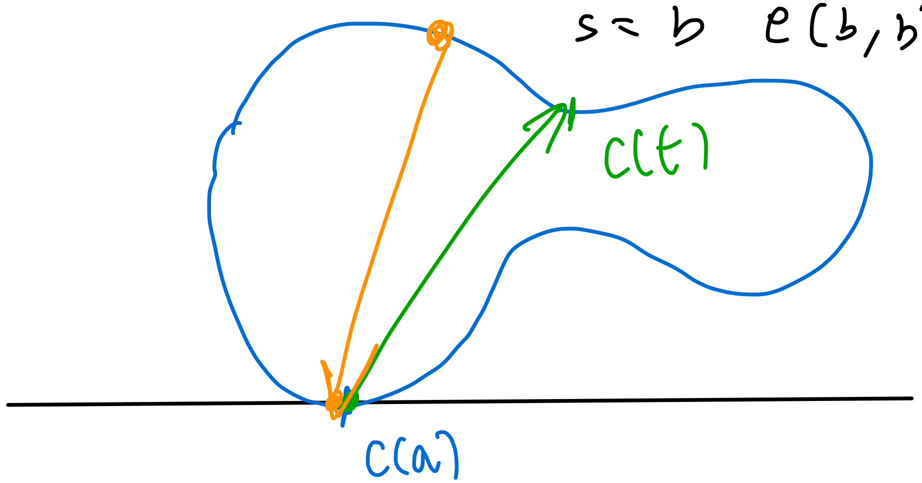
$$e\big|_{s=t} = \frac{c'(t)}{|c'(t)|}$$

$$\Big[\theta(s,b)\Big]_{s=a}^{s=b} \qquad \Big[\theta(a,t)\Big]_{t=a}^{t=b}$$

<u>trick</u> $\theta(b,b) - \theta(a,a) = \overbrace{\theta(b,b) - \theta(a,b)}^{\pi} + \overbrace{\theta(a,b) - \theta(a,a)}^{\pi}$

$s = a : e(a,b) = (-1,0)$

$s = b \quad e(b,b) = (+1,0)$



$$\boxed{e(a,t) \in S^1 \cap \{x^2 \geq 0\}}$$

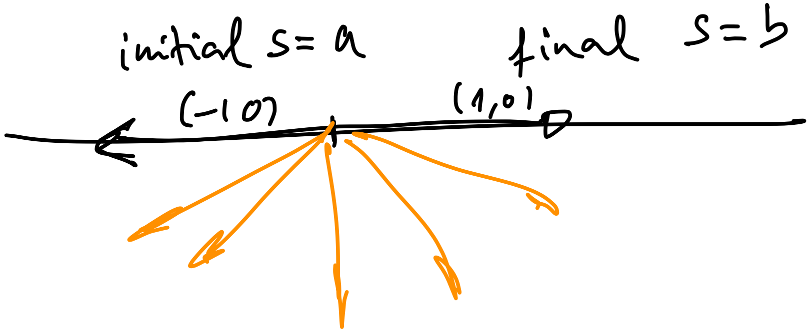
$e(a,a) = (1,0)$

$e(a,b) = (-1,0)$

differne in angle $= \pi$

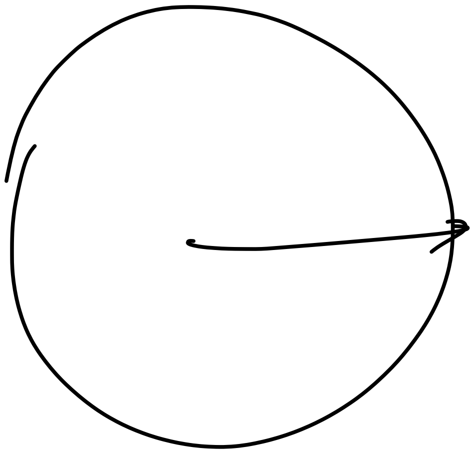
$$\boxed{e(s,b) \in S^1 \cap \{x^2 \leq 0\}}$$

difference in angle $\theta = \pi$



$$\Rightarrow \quad \theta(b,b) - \theta(a,a) = \pi + \pi = 2\pi$$

S^1



1.7 Thm (Fenchel 1923 $(n=3)$, Borsuk 1947)

If $c : [0,L] \longrightarrow \mathbb{R}^3$ C^2 closed unit speed curve,

[Assume $c([0,L])$ is not contained in some plane]

Then $\int_0^L \kappa(s)\, ds \geq 2\pi$

$>$

$$\frac{d}{dt}\ c' \cdot c' = 1$$
$$\Rightarrow 2c'' \cdot c' = 0$$

Remark $|c'| = 1$ $\kappa = c'' \cdot e_2 = |c''|$

Therefore, $\int_0^L \kappa(s)\, ds = \int_0^L |c''|\, ds$

$$= \int_0^L |(c')'|\, ds = L(c')$$

$c' : [0, L] \longrightarrow S^2 \subset \mathbb{R}^3$ <u>tangent indicatrix</u>

In order to prove thm 1.7 <u>2 steps</u>

(1) Show $c'([0,L]) \not\subset$ any hemisphere

(2) Use the following proposition (to $c \leftarrow c'$)

<u>Prop 1.8</u> $c : [a,b] \longrightarrow S^2 \subset \mathbb{R}^3$ is a <u>closed</u> curve and $c([a,b]) \not\subset$ hemisphere

$$L(c) \geq 2\pi$$

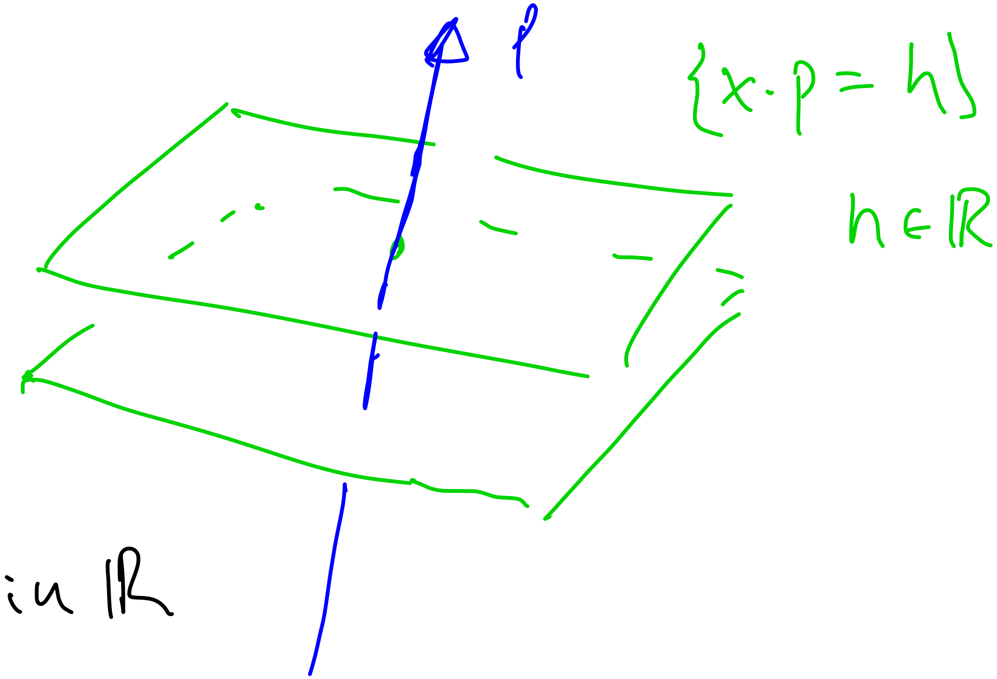
(1) $\forall\, p \in S^2$ hemisphere $\{x \in S^2 : x \cdot p \geq 0\} =: H_p$

If $\exists\, p$ st. c' contained in H_p

$$c' \cdot p \geq 0 \iff (c \cdot p)' \geq 0$$

$$t \longmapsto c(t) \in \mathbb{R}^3$$

$$t \longmapsto (c \cdot p)(t)$$



since c is closed

$\Rightarrow$ $c \cdot p = h$ for some h in $\mathbb{R}$

$c : [0, L] \to S^2$

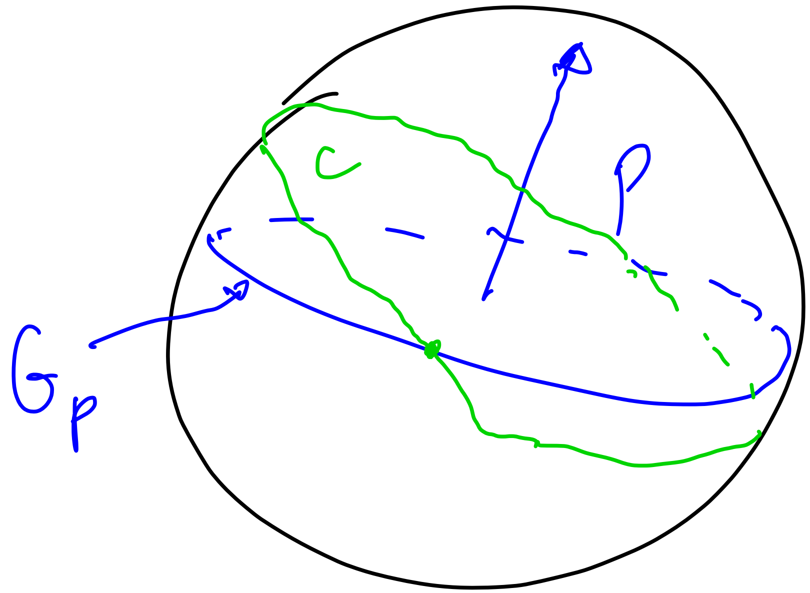
p uniformly distributed in S^2

$$L(c) = \pi \, \mathbb{E}\left(\# \, c([0,L]) \cap G_p \right)$$

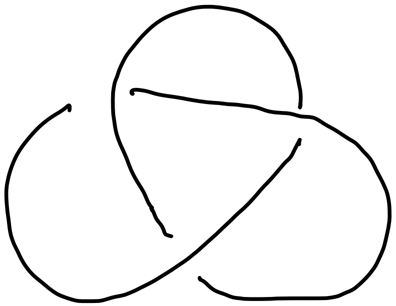
$$G_p = \{ x \in S^2 : x \cdot p = 0 \}$$

Curve c never contained in $\{ x \cdot p \geq 0 \}$

$\Rightarrow \quad \forall p \quad \underline{c}$ intersects G_p

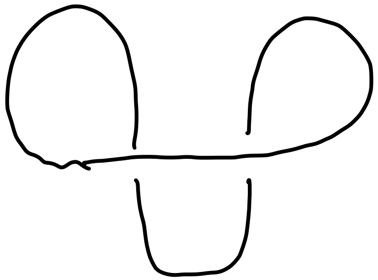


19. Thm If $c : [0, L] \longrightarrow \mathbb{R}^3$ is a closed C^2 Knoted curve. Then $\int_0^L \kappa(s)\, ds > 4\pi$

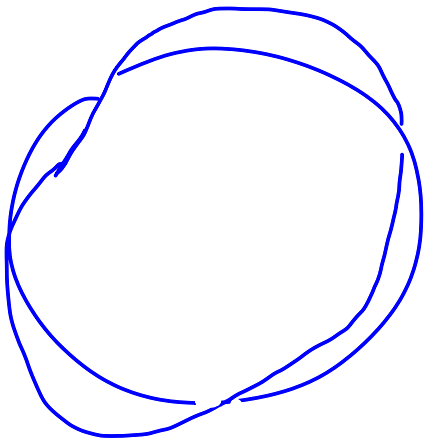


Knoted

trefoil Knot



unknoted



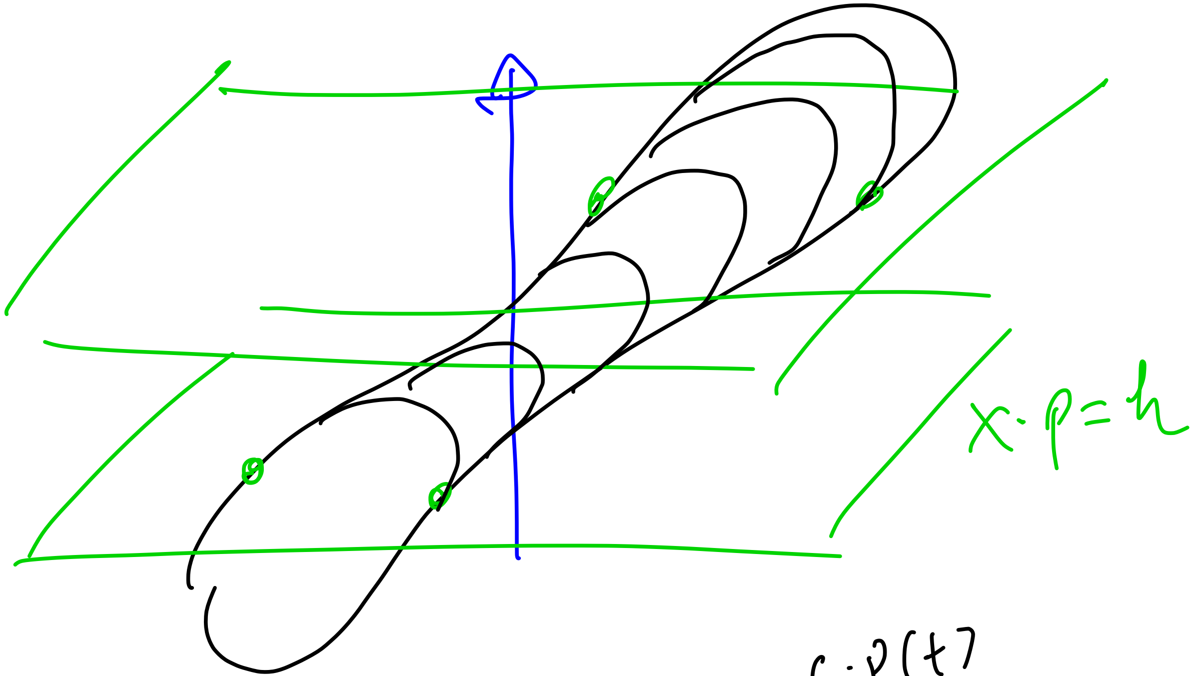
Fáry 1949

Milnor 1950

C knoted $\Rightarrow$ $\forall\, p \in S^2$ $\exists\, h \in \mathbb{R}$ "height"

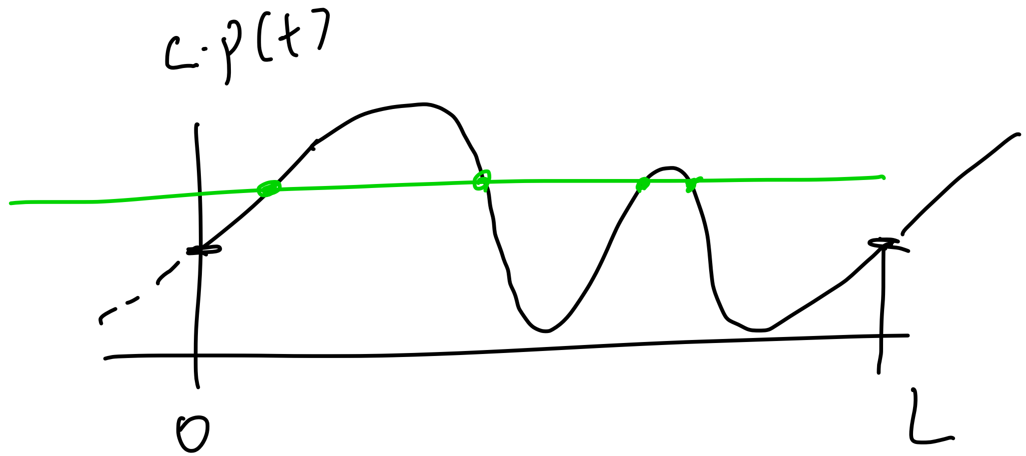
for which

$$\#\, c([0,L]) \cap \{x \in \mathbb{R}^3 : x \cdot p = h\} \geq 4$$



$[0,L] \longmapsto c \cdot p$

is $[0,L]$ C^2 periodic



$$\exists\, h : \quad \{ c \cdot p = h \} \geq 4$$

$$\Longrightarrow \quad \{ c' \cdot p = 0 \} \geq 4 \quad \text{for all } p$$

$$\Longrightarrow \quad L(c') = \pi\, \mathbb{E}\Big(\underbrace{c'([0,L]) \cap G_p}_{4} \Big)$$

$$\geq 4\pi$$

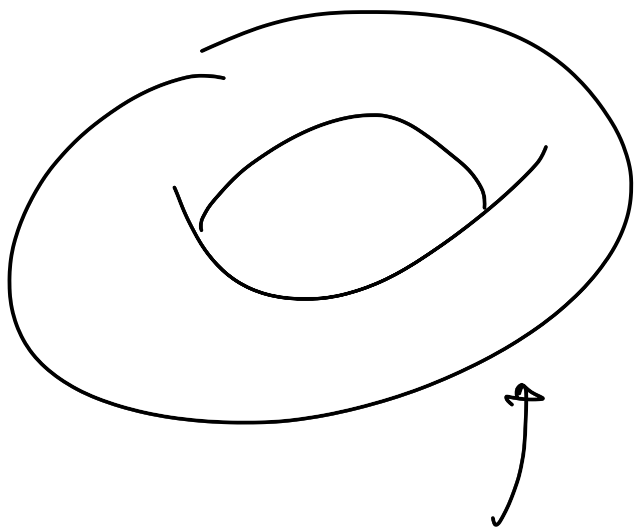
$\blacksquare$

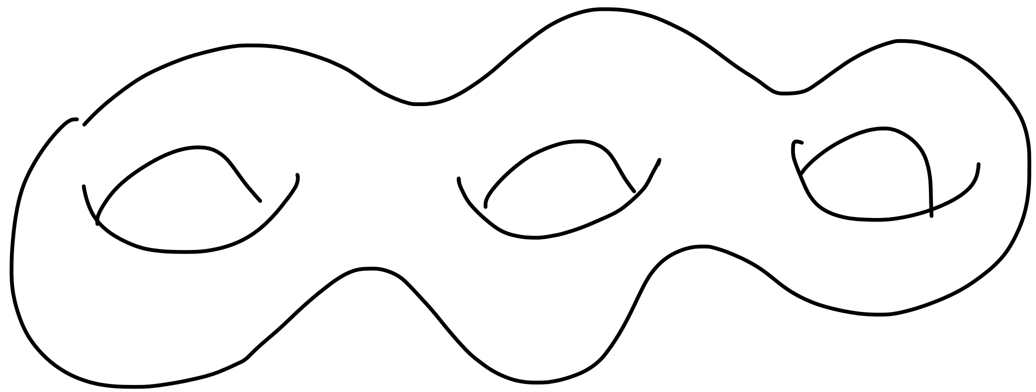
Ch 2 "Surfaces"

$F(x) = |x|^2 - 1$

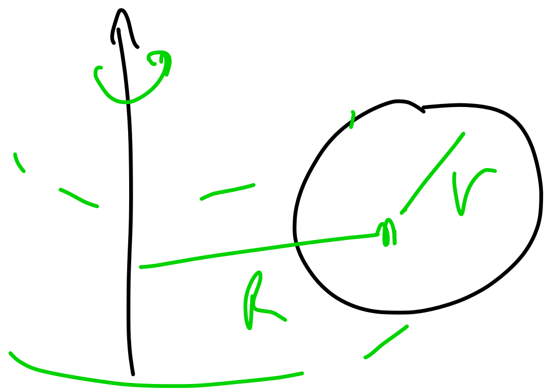
$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x|^2 = \sum (x^i)^2 = 1\} \subset \mathbb{R}^n$$

$$T^2 = S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$$ Clifford-Torus





parametrization of torus



$$\big(\cos\theta(R + r\cos\phi),\ \sin\theta(R + r\cos\phi),\ r\sin\phi\big)$$

$\theta \in (0, 2\pi) \quad \phi \in (0, 2\pi)$

soap film / soap bubble

→ Why are soap bubbles spherical ?

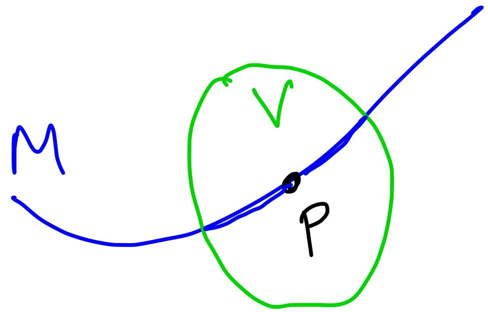
Submanifolds and immersions

<u>2.1 Def</u> A set $M \subset \mathbb{R}^n$ is m-dim. submanifold of $\mathbb{R}^n$ if $\forall p \in M$ $\exists$ open nbhd $V \subset \mathbb{R}^n$ of p and C^∞ diffeo $\varphi : V \to U$ onto open set $U \subset \mathbb{R}^n$ s.t

$$\varphi(M \cap V) = (\mathbb{R}^m \times \{0\}) \cap U$$

$$(\underbrace{0, \dots, 0}_{n-m})$$

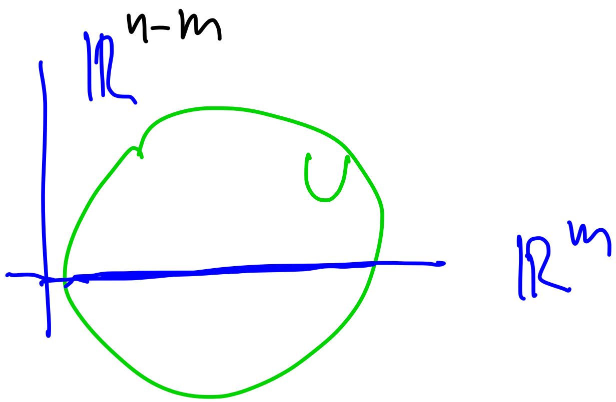
$\mathbb{R}^n$ (ambient)



$\xrightarrow{\varphi}$

<u>submanifold chart</u>

(Schnittkante)



$k := n - m \geq 1$ codimension of M in $\mathbb{R}^n$

$W \subset \mathbb{R}^n$ open, $F : W \to \mathbb{R}^k$ differentiable $k \leq n$

$p \in W$ is a $\begin{cases} \underline{\text{regular pt.}} \text{ of } F \text{ if } dF_p \text{ is surjective} \\ \underline{\text{singular pt}} \text{ otherwise} \end{cases}$

$x \in \mathbb{R}^k$ is a $\begin{cases} \underline{\text{regular}}\ \underline{\text{value}} \text{ of } F \text{ if all } p \in F^{-1}\{x\} \text{ are regular} \\ \underline{\text{singular}}\ \underline{\text{value}} \text{ otherwise} \end{cases}$

<u>2.2 Thm</u> (reg. value thm) $W \subset \mathbb{R}^n$ open, $F : W \to \mathbb{R}^k$, C^∞, $x \in \mathbb{R}^k$ regular value of F

$\Longrightarrow$ $M := F^{-1}\{x\}$ is a submfld of $\mathbb{R}^n$

of dimension $m := n - k$

(so k is the codimension)

pf. Let $p \in M = F^{-1}\{x\}$ (nonempty!);

$\Rightarrow$ p is a regular pt. [suppose w.l.o.g. $x = 0$]

$F \leftarrow F - x$

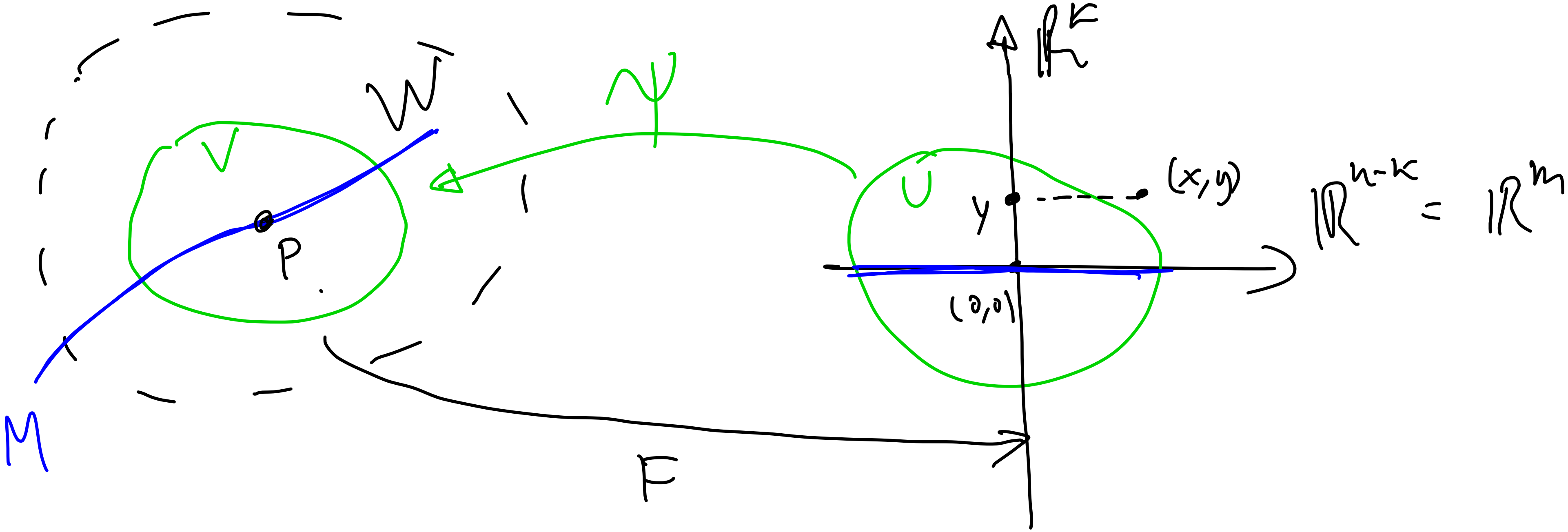
apply the following:

thm (Appendix A2 lecture note of Prof. Lang) Implicit fc'n thm in surjective form

$W \subset \mathbb{R}^n$ open, $F: W \to \mathbb{R}^k$ C^∞ $p \in W$, $F(p) = 0$

dF_p surjective $\Rightarrow$ $\exists$ open nbhds $U \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ of $(0,0)$ and $V \subset W$ of p, and C^∞-diffeo $\psi: U \to V$

s.t $\psi(0,0) = p$ and $(F \circ \psi)(x,y) = y$ $\forall (x,y) \in U$



$M = F^{-1}\{0\}$

$\varphi := \psi^{-1} : V \to U$ is a submfld chart around p

Notice $\varphi(M \cap V) = \{ (x,y) \in U \mid \psi(x,y) \in M = F^{-1}\{0\} \}$

$$\Updownarrow$$

$$\underbrace{F \circ \psi}_{y}(x,y) = 0$$

Examples $SL(n,\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid \det A = 1 \} \subset \mathbb{R}^{n\times n} = \mathbb{R}^{n^2}$

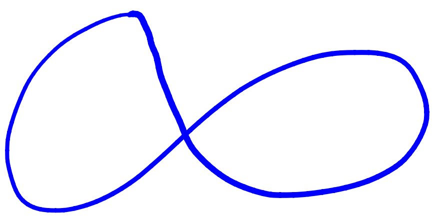
$O(n,\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid AA^T = Id_n \}$

$SO(n,\mathbb{R})$ = submanifold of $\mathbb{R}^{n\times n}$

2.3. Def'n $U \subset \mathbb{R}^m$ open, $n > m$. A C^∞-map $f : U \to \mathbb{R}^n$ is a *regular m-dim surface* or an *immersion* if

df_x is *injective* $\forall x \in U$

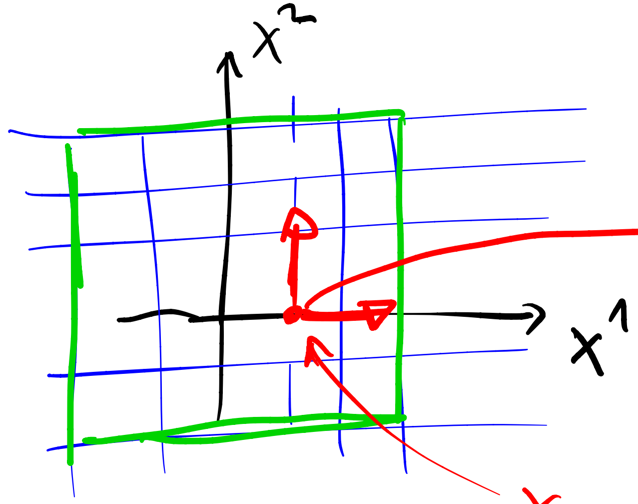
(Remark: f itself need not be injective, like for regular curves)

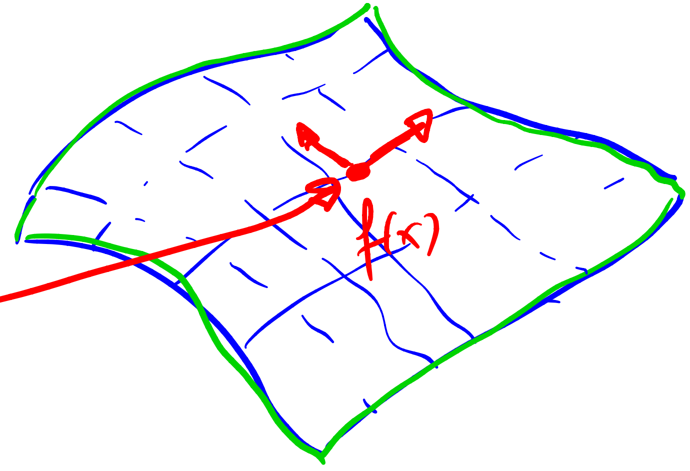


Picture of 2D parametrized surface

$\mathbb{R}^2$ $(m=2)$ $\qquad\qquad\qquad\qquad$ $\mathbb{R}^3$

$$f = (f^1, f^2, f^3)$$





df_x

df_x injective

$$\frac{\partial f}{\partial x^1} = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} \\ \frac{\partial f^2}{\partial x^1} \\ \frac{\partial f^3}{\partial x^1} \end{pmatrix}$$

these two vectors be linearly indep $\forall x \in U$

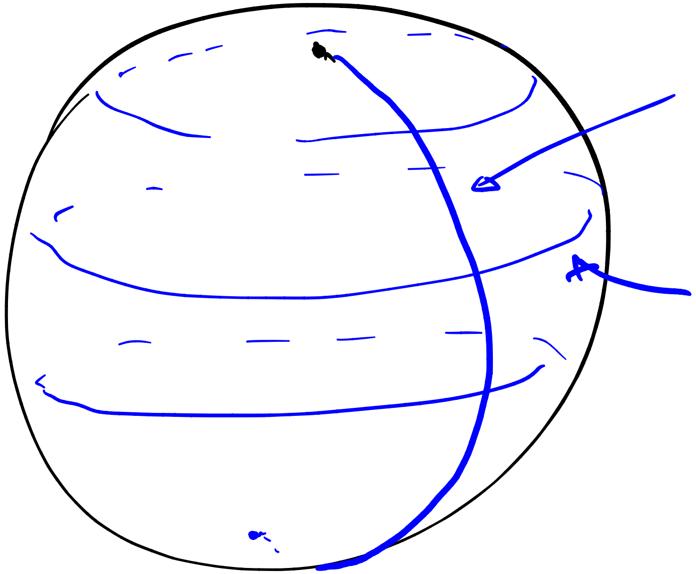
same with $\frac{\partial f}{\partial x^2} = \begin{pmatrix} \\ \end{pmatrix}$

Examples

1. regular curves $\quad c : \overset{\text{open}}{I} \longrightarrow \mathbb{R}^n \quad$ 1-d immersion

2. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \quad$ (x,y) instead (x^1, x^2)

$$f(x,y) = (\cos x \cos y,\ \sin x, \sin y,\ \sin y)$$



$$f\big|_{(0,2\pi)\times(-\pi/2,\pi/2)}$$

$$f\big|_{\mathbb{R}\times(-\pi/2,\pi/2)}$$

Observation $\quad df_{(x,y)}$

(exercise)

$$\frac{\partial f}{\partial x} = 0 \quad \text{when } \cos y = 0$$

$$\iff y = \pm \frac{\pi}{2}$$

$\iff$ north and south pole

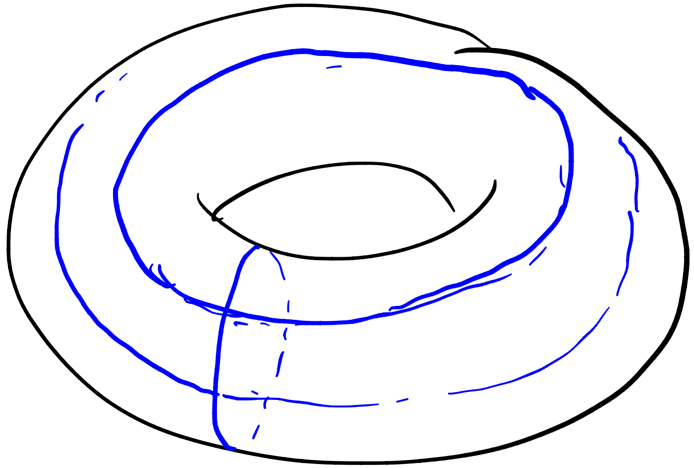
3. $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = \big((R + r\cos y)\cos x,\ (R + r\cos y)\sin x,\ r\sin y\big)$$

$(x,y) \in \mathbb{R}^2$

$R > r > 0$

regular immersion



2.4. Thm (Immersion thm) $U \subset \mathbb{R}^m$ open, $f \in C^\infty(U, \mathbb{R}^n)$ immersion $\Rightarrow$ $\forall x \in U$ $\exists$ open nbhd $U_x \subset U$ of x s.t. $f(U_x)$ is an m-dim submanifold of $\mathbb{R}^n$

proof suppose $x = 0 \in U$ (replace f by $f(\cdot - x)$

$f(0) = p$, and apply

see lect notes Appendix 3

Implicit fc'n thm (injective form): $U \subset \mathbb{R}^m$ open

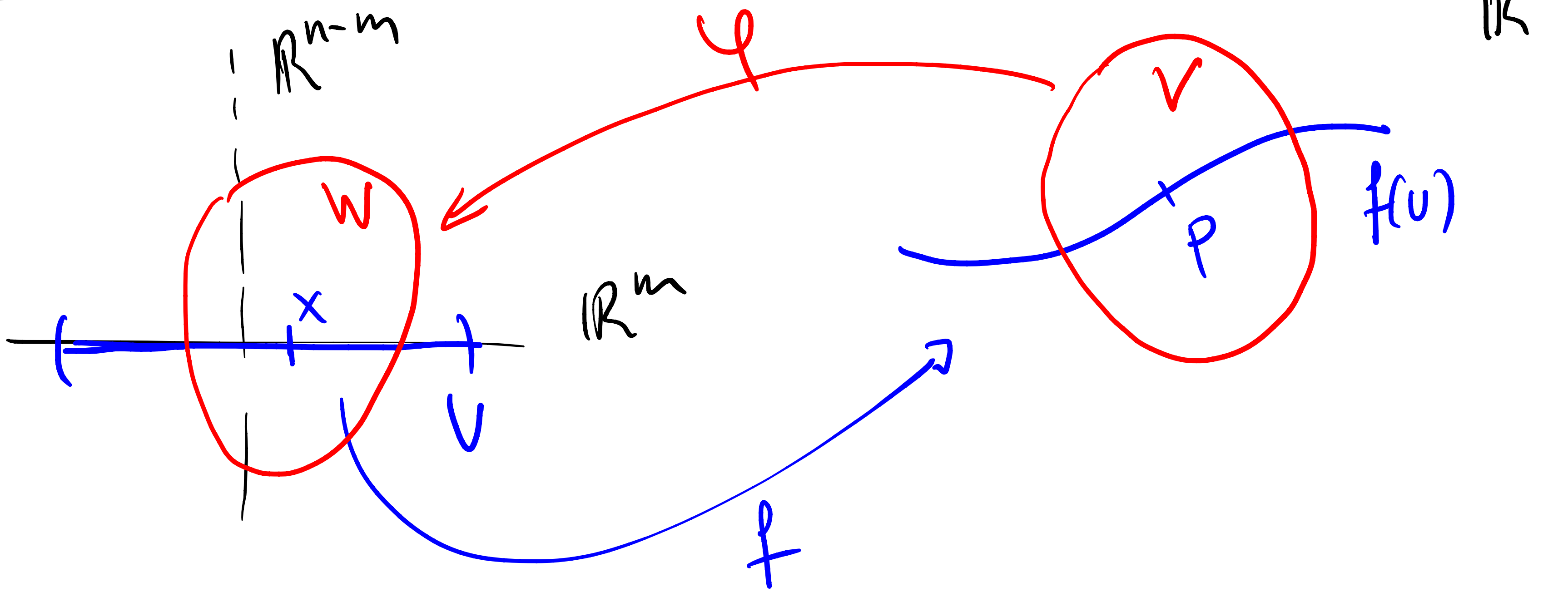
$f: U \to \mathbb{R}^n$, C^∞, $0 \in U$, $f(0) = p$, df_0 injective

$\Rightarrow$ $\exists$ open nbhds $V \subset \mathbb{R}^n$ of p and $W \subset U \times \mathbb{R}^{n-m}$ of $(0,0)$, and a C^∞-diffeo $\varphi: V \to W$

s.t. $\varphi(p) = (0,0)$ and

$$(\varphi \circ f)(x) = (x, 0) \qquad \forall (x,0) \in W$$

Drawing of thm



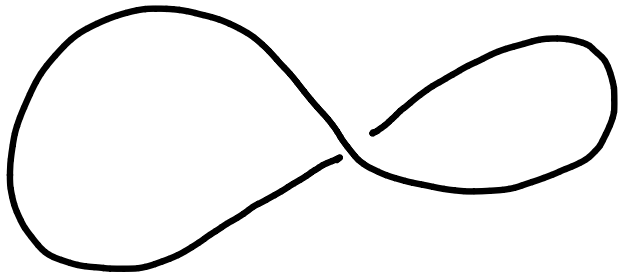
Applying this, we immedictely have thm 2.4

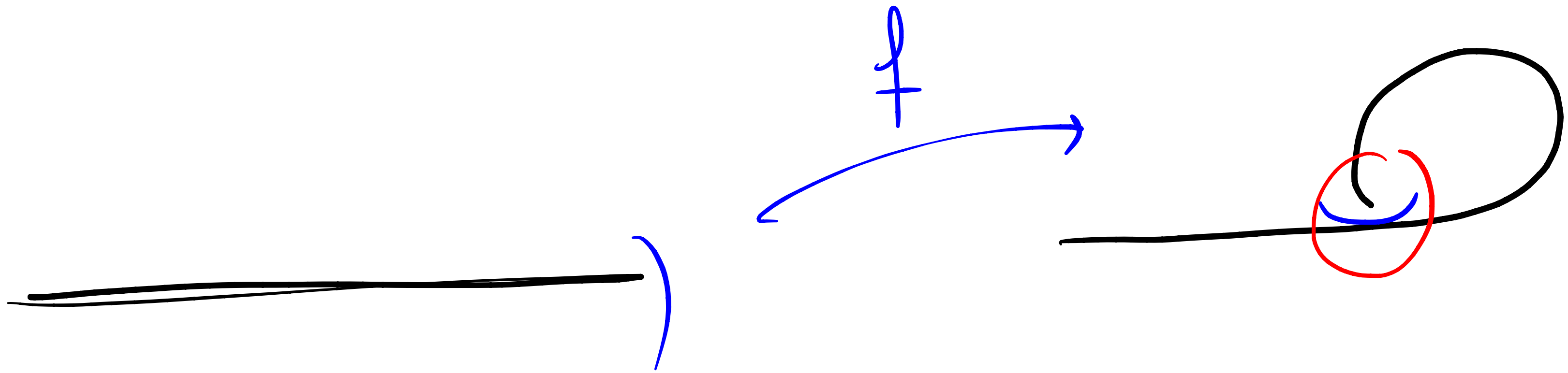
φ is the submanifold chart around $x=0$

$\blacksquare$

Remark The image of an injective immersion $f: U \to \mathbb{R}^n$ need not be a submanifold

$$c : (0, 2\pi) \to \mathbb{R}^2 \qquad c(t) = (\sin(t), \sin(2t))$$





positive result

2.5. thm (local parametrizations)

the following two are equivalent, for $M \subset \mathbb{R}^n$

(i) M is an m-dim submfld (defn 2.1)

(ii) $\forall p \in M$ $\exists$ open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ and an immersion $f : U \to \mathbb{R}^n$ s.t $p \in f(U) = M \cap V$, and $f : U \to M \cap V$ is a homeomorphism

proof (i) $\Rightarrow$ (ii)

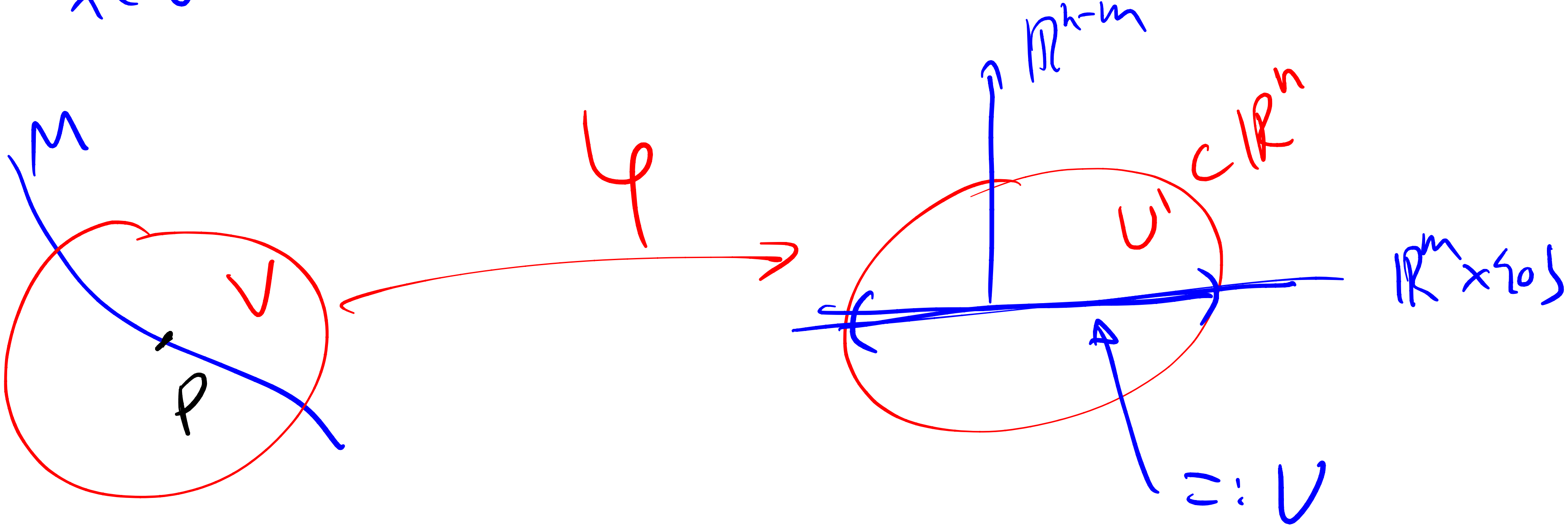
take $p \in M$, $\varphi : V \to U' \subset \mathbb{R}^n$ submanifold chart (here V is a nbhd of p), by defn 2.1

Put $U := \{ x \in \mathbb{R}^m : (x, 0) \in U' \}$

$f(x) := \varphi^{-1}(x, 0)$, then $f(U) = \varphi^{-1}((\mathbb{R}^m \times \{0\}) \cap U')$

$= M \cap V$

$x \in U$



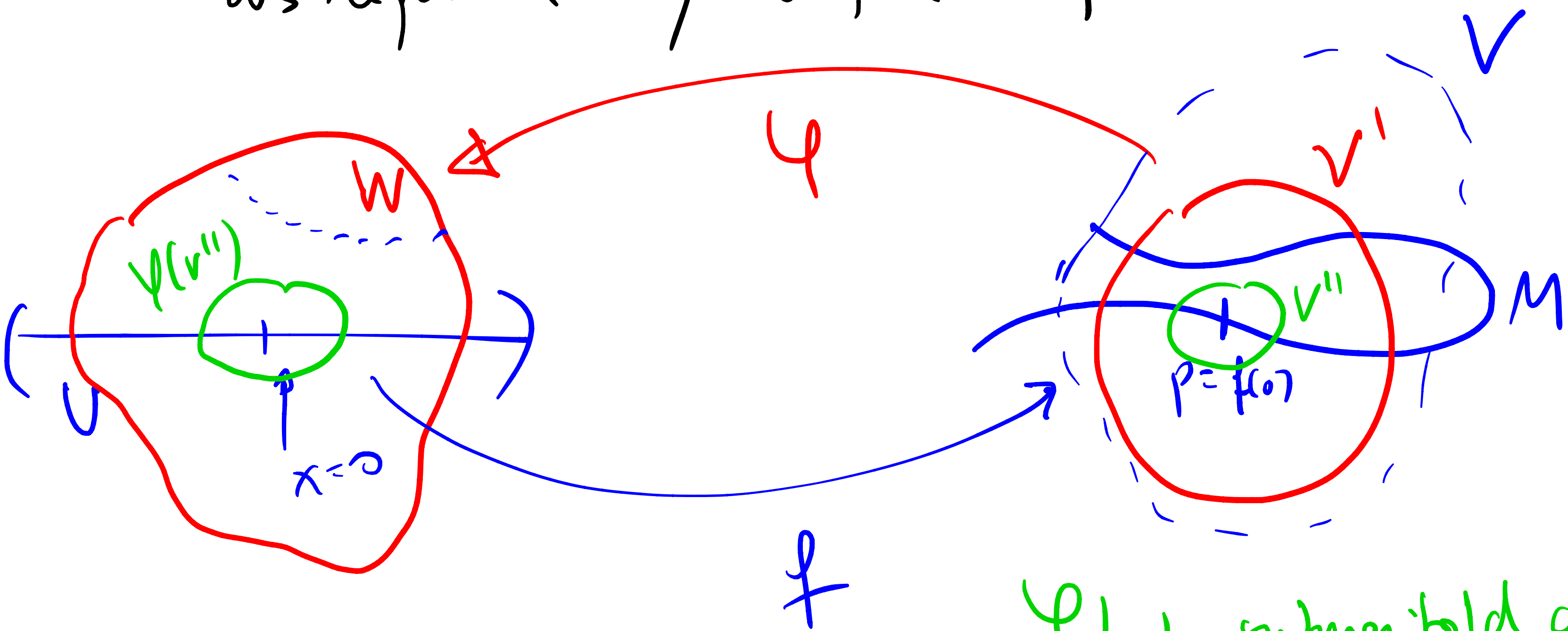
(ii) $\Rightarrow$ (i)

Take $p \in M$, by assumption $\exists\, U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ open

$f : U \to \mathbb{R}^n$ immersion s.t $f(0) = p$,

f homeo. onto $M \cap \underline{\underline{V}}$

<u>Goal</u>: Construct submanifold chart φ around p, as required by def'n 2.1



$\varphi|_{V'}$ submanifold chart ✓

Apply implicit fc'n thm (injective form)

$\Rightarrow$ $\exists$ open nbhds $V' \subset V$ of p and $W \subset U \times \mathbb{R}^{n-m}$ of $(0,0)$, and C^∞ diffeo $\varphi : V' \to W$ s.t.

$\varphi(p) = (0,0)$ and $(\varphi \circ f)(x) = (x, 0)$ $\forall (x,0) \in W$

$(\forall x \in \varphi(W) \cap U)$

Warning: in general

$$f^{-1}(M \cap V') \not\subset \{x \mid (x,0) \in W\}$$

so, I need to restrict domains (using the homeomorphism hyp.)

$f^{-1} : M \cap V \to U$ continuous, so

$\exists$ open nbhd $V'' \subset V'$ of p st. $U_0 := f^{-1}(M \cap V'')$

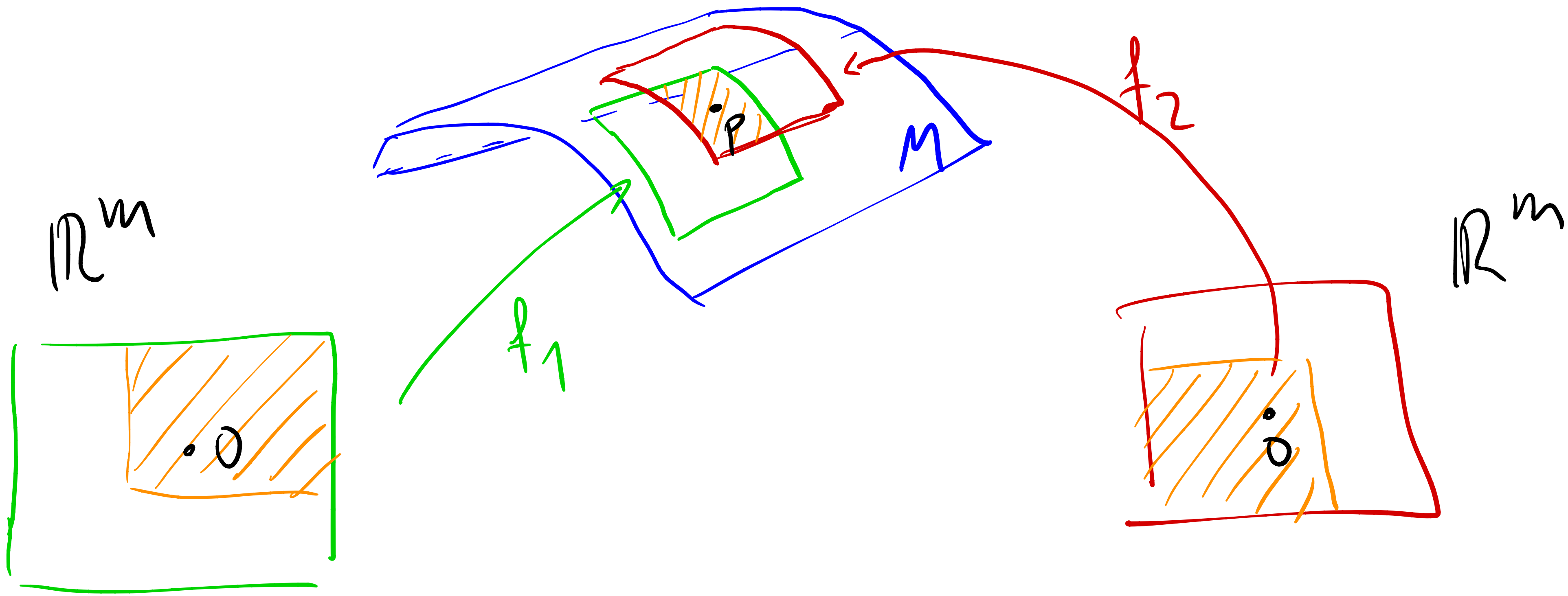
$\subset \{x \mid (x,0) \in W\}$

Therefore $\varphi|_{V''}$ is a submanifold chart

2.6 Lemma (param. transformation / change of charts)

$M \subset \mathbb{R}^n$ m-dim submanifold, $f_i : U_i \to f(U_i) \subset M$, $i = 1, 2$ are two local param. with $V := f_1(U_1) \cap f_2(U_2) \neq \emptyset$

$\Rightarrow$ $\Psi := f_2^{-1} \circ f_1 : f_1^{-1}(V) \to f_2^{-1}(V)$ is C^∞ diffeo



<u>proof</u> Suppose w.l.o.g. $f_1(0) = f_2(0) = p$

Exactly as in the pf. of thm 2.4, $\exists$ diffeo φ defined in nbhd of p in $\mathbb{R}^n$ with $\varphi(p) = (0,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$

s.t. (#) $\boxed{\varphi(f_2(x)) = (x,0)}$ $\forall (x,0) \in \text{image}(\varphi)$

$$\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^m \quad \text{canonical projection}$$
$$(x,y) \longmapsto x$$

then $$\psi = f_2^{-1} \circ f_1 = \pi \circ \varphi \circ f_1 \in C^\infty$$

Since f_1, f_2 play symmetric role also

$$\psi^{-1} = f_1^{-1} \circ f_2 \text{ is } C^\infty$$

∎

$$\left[\begin{array}{ll} \pi \circ \varphi \circ f_2 = \mathrm{id} & \circ \psi = f_2^{-1} \circ f_1 \text{ both sides} \\ \pi \circ \varphi \circ f_1 = \psi & \end{array}\right]$$

2.7 Def The *tangent space* TM_p of an m-dim. submfld $M \subset \mathbb{R}^n$ at $p \in M$ is $TM_p := df_x(\mathbb{R}^m) \subset \mathbb{R}^n$ for some (and every!) local param $f : U \to f(U) \subset M$ with $f(x) = p$

exercise use Lemma 2.6 and the standard chain rule to see that this is independent of the f we choose

The *normal space* $TM_p^\perp$ is the orthogonal complement of TM_p inside $\mathbb{R}^n$

Since f is immersion, TM_p is a m-dim linear space and $TM_p^\perp$ is K-dim, $K = n-m$ codimension

2.8 Def A map $F: M \longrightarrow \mathbb{R}^l$, $M \subset \mathbb{R}^n$ submfld, is <u>differentiable</u> at $p \in M$, if for some (hence every!) local param. $f: U \to f(U) \subset M$ with $f(x) = p$ the composition

$$F \circ f : U^{\subset \mathbb{R}^m} \longrightarrow \mathbb{R}^l$$

is differentiable at $x \in U$.

The <u>differential of F</u> at p is the unique linear map

$$dF_p : TM_p \longrightarrow \mathbb{R}^l$$

st. the following holds for some (hence every) loc. param. f

$$\underbrace{d(F\circ f)_x}_{\checkmark} = \underbrace{dF_p}_{\text{not yet defined}} \circ \underbrace{df_x}_{\checkmark} \qquad \text{"Chain rule"}$$

exercise: check this def'n is independent of the local parametrization used

take two local param. $f, \tilde{f}$ s.t. $f(x) = \tilde{f}(y) = p$

$\tilde{f} = f \circ \psi$ $\quad \psi$ diffeo $\quad \underline{\psi = f^{-1} \circ \tilde{f}}$ restricted to "common domain" (as in lemma 2.6)

$$d(F\circ f)_x = A\, df_x$$

$$d(F\circ \tilde{f})_y = \tilde{A}\, d\tilde{f}_y \qquad \Rightarrow \qquad A = \tilde{A}$$

Use chain rule you know

$$(F \circ f \circ \Psi) = F \circ \tilde{f} \qquad (\Psi(y) = x)$$

$$\left.\begin{array}{l} d(F\circ f)_x \, d\Psi_y = d(F \circ \tilde{f}) \\ \quad \| \qquad\qquad\qquad\qquad \| \\ A\, \underbrace{df_x \circ d\Psi_y}_{d(f\circ\Psi)_y = d\tilde{f}_y} \qquad \tilde{A}\, d\tilde{f}_y \end{array}\right\} \Rightarrow A \circ \cancel{d\tilde{f}_y} = \tilde{A} \circ \cancel{d\tilde{f}_y}$$

isomorphism!

$$\Rightarrow A = \tilde{A}$$

Remark Useful fact about dF_p $\quad v = C'(0) \in TM_p$

for some curve $C: (-\varepsilon, \varepsilon) \to f(U) \subset M$ with $C(0) = p$

then $\exists\ \gamma: (-\varepsilon, \varepsilon) \to U$ s.t $\quad C = f \circ \gamma$

$$v = C'(0) = df_{\gamma(0)}(\gamma'(0))$$

$$\Rightarrow \quad dF_p(v) = (dF_p \circ df_{\gamma(0)})(\gamma'(0))$$

def'n of dF_p

$$= d(F \circ f)_{\gamma(0)}(\gamma'(0))$$

$$= (F \circ f \circ \gamma)'(0)$$

$$= (F \circ c)'(0)$$

curve $(-\varepsilon, \varepsilon) \to \mathbb{R}^l$

Orientability and the separation thm

2.8 Def'n A submanifold $M \subset \mathbb{R}^n$ is orientable if $\exists$ "system" $\{f_\alpha : U_\alpha \to f_\alpha(U_\alpha) \subset M\}_{\alpha \in A}$ of local param.

s.t. $\bigcup_{\alpha \in A} f(V_\alpha) \supset M$ such that for every parameter transformation $f_\beta^{-1} \circ f_\alpha$ with $\alpha, \beta \in A$ and $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \neq \emptyset$ the orientation is preserved: that is

at any point

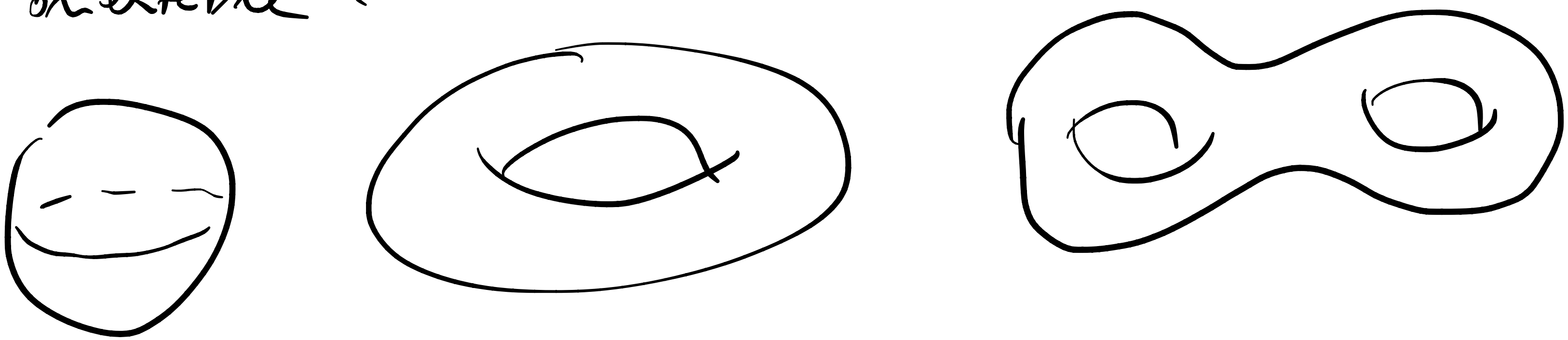
$$d(f_\beta^{-1} \circ f_\alpha)_x \quad \text{\underline{positively oriented}}$$

this is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, we are saying it must have > 0 determinant

---

<u>Examples</u>

1. compact connected submanifolds of co-dim 1 are orientable (2-10)



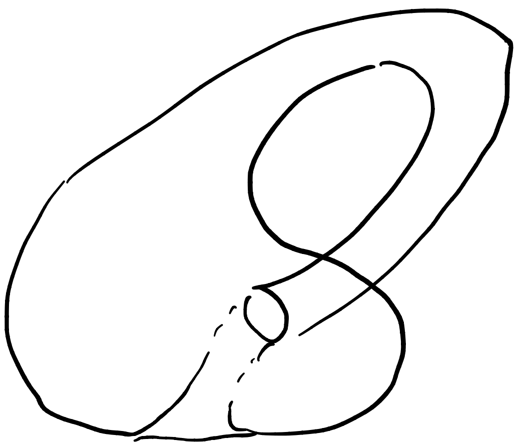
2. Möbius band embed it in $\mathbb{R}^3$ as you like

not orientabl

3. "Klein bottle"

- immersed in $\mathbb{R}^3$
- embedded in $\mathbb{R}^4$

not orientabe (compact)



2.9 Lemma An m-dim submfld $M \subset \mathbb{R}^{m+1}$ is orientable

$\iff$ $\exists$ "cont. unit normal vector field"

$$N : M \longrightarrow S^m \quad \text{s.t} \quad N(p) \in TM_p^{\perp} \quad \forall p \in M$$

proof

$\Rightarrow$ | M is orientable, $\exists \{f_\alpha : U_\alpha \to M\}_{\alpha \in A}$

s.t $\bigcup_{\alpha \in A} f_\alpha(U_\alpha) = M$, with $\det(f_\beta^{-1} \circ f_\alpha) > 0$

Given α define $\nu_\alpha : U_\alpha \longrightarrow S^m$ "along f_α"

with $\nu_\alpha(x) \in TM^{\perp}_{f_\alpha(x)}$ for all x

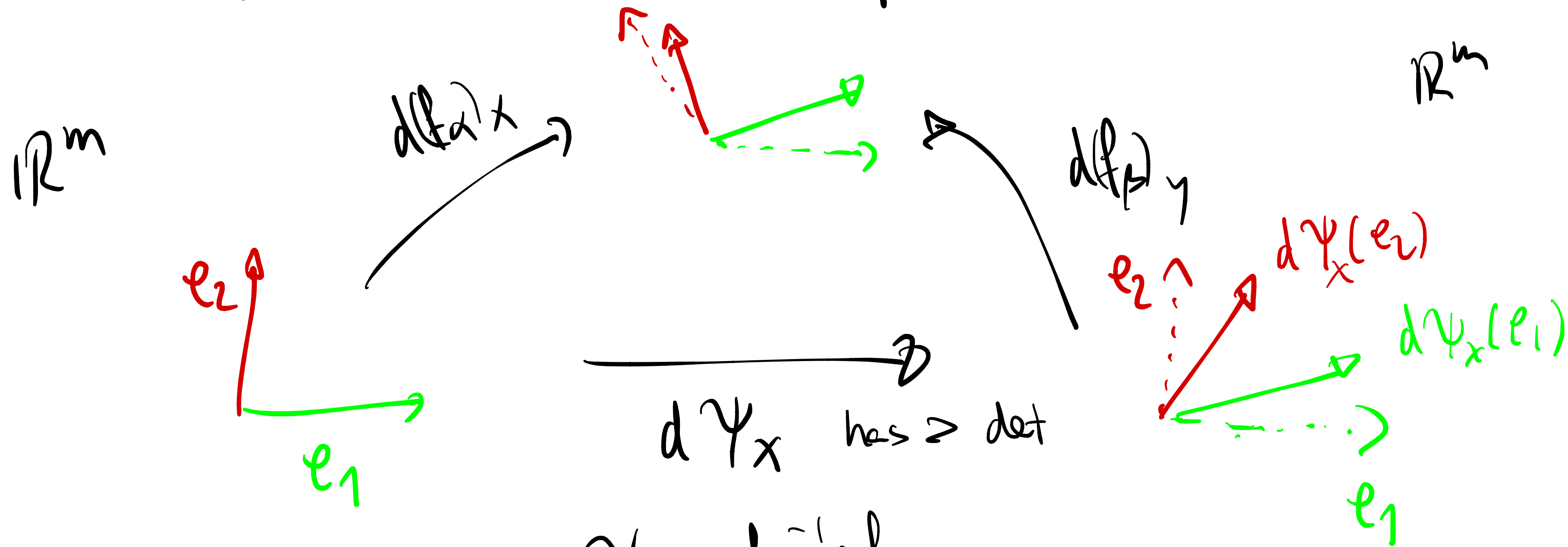
s.t $\dfrac{\partial f_\alpha}{\partial x^1}, \ldots, \dfrac{\partial f_\alpha}{\partial x^m}, \nu_\alpha$ is positive basis of $\mathbb{R}^{m+1}$

We must show $\left[f_\alpha(x) = f_\beta(y) = p \overset{(*)}{\Longrightarrow} \nu_\alpha(x) = \nu_\beta(y) \right]$

$N(p) := \nu_\alpha(x)$ for some α s.t. $x \in U_\alpha$, $f_\alpha(x) = p$

is giving a consistent (well-defined) unit normal vector field

TM_p



$\Psi = f_\beta^{-1} \circ f_\alpha$

Since the two basis of TM_p have the same orientation the sign I need to give to $V(p)$ so that it completes them to a > basis of $\mathbb{R}^{m+1}$ is the same in the two cases

In $\mathbb{R}^3$,

$$\frac{\frac{\partial f_\alpha}{\partial x^1} \times \frac{\partial f_\alpha}{\partial x^2}}{\left| \quad \right|} \quad \text{is } C^\infty(U_\alpha)$$

$$\det\left(\frac{\partial f_\alpha}{\partial x^1}, \frac{\partial f_\alpha}{\partial x^1}, N\right) = \left(\frac{\partial f_\alpha}{\partial x_1} \times \frac{\partial f}{\partial x^2}\right) \cdot N$$

$\Leftarrow$ Conversely, if N is given, let us contruct a system of param with > 0 paramet tranformations.

Chose $\{f_\alpha : U_\alpha \longrightarrow M\}_{\alpha \in A}$ s.t $\bigcup_{\alpha \in A} f(U_\alpha) = M$

and $\frac{\partial f_\alpha}{\partial x^1}, \dots \frac{\partial f_\alpha}{\partial x^m}, N$ is $\underbrace{>0}$ oriented

$\left[\text{if } \underbrace{<0} \text{ replace } f_\alpha \text{ by } f_\alpha(-x_1, x_2, \dots x_m) \right]$

with this choice the det of parameter transformations will be always >0 exactly as before. ∎

2.10 Thm (Separation thm)

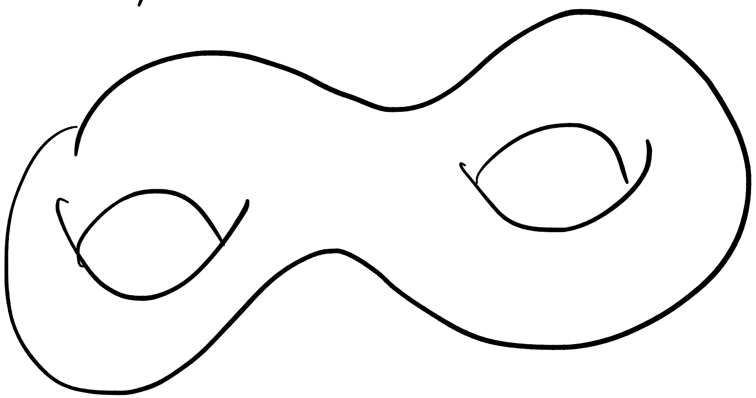
$\emptyset \neq M \subset \mathbb{R}^{m+1}$ m-dim compact connected submanifold

$\Longrightarrow$ $\mathbb{R}^{m+1} \setminus M$ has exactly 2 connected components,

A, B st $M = \partial A = \partial B$,

and M is orientable

(Moreover, one of the components is bdd and the other unbounded)

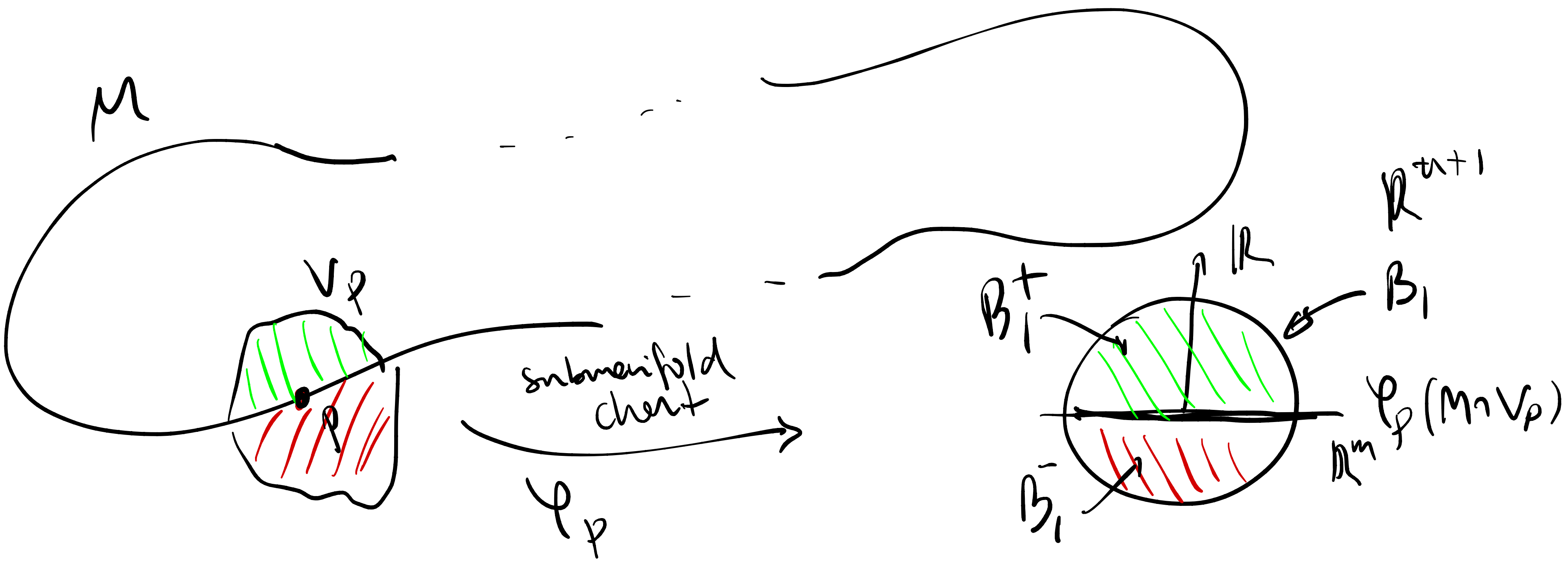


plan of the proof

Step 1 $\forall p \in M$ $\exists$ open nbhd V_p of p in $\mathbb{R}^{m+1}$

s.t
- $V_p \setminus M$ has 2 connected comp V_p^+, V_p^-
- $\exists$ 2 different connected comp $\mathbb{R}^{m+1} \setminus M$, A_p, B_p

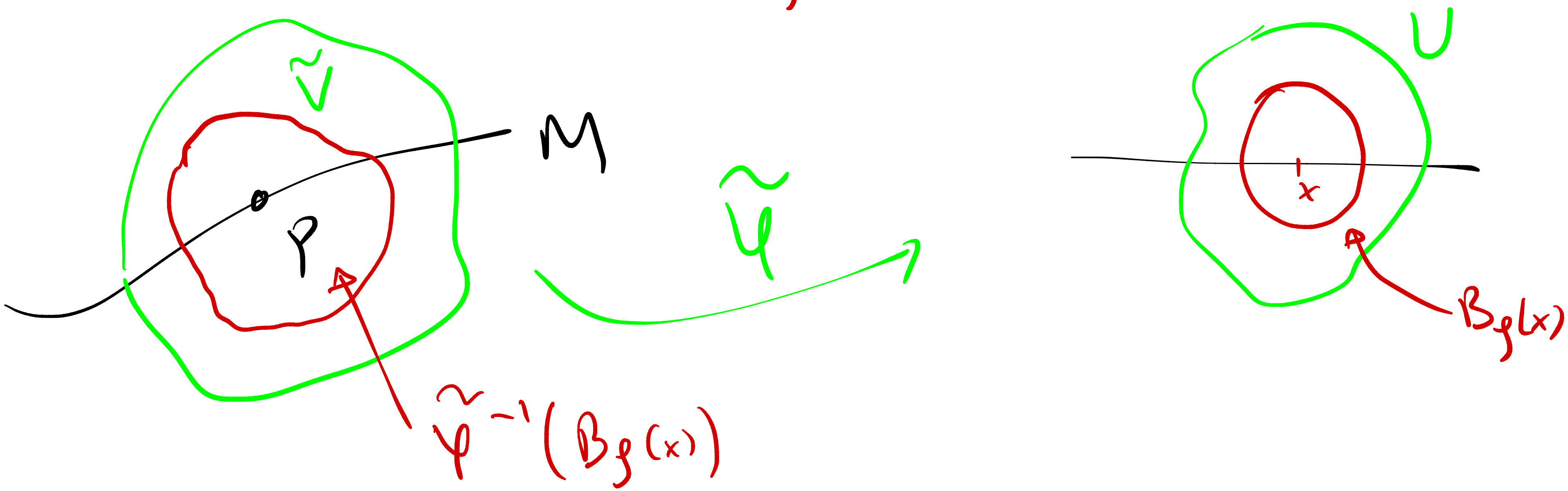
  s.t $V_p^+ \subset A_p$, $V_p^- \subset B_p$



Remark by def'n a submanifold chart is a map

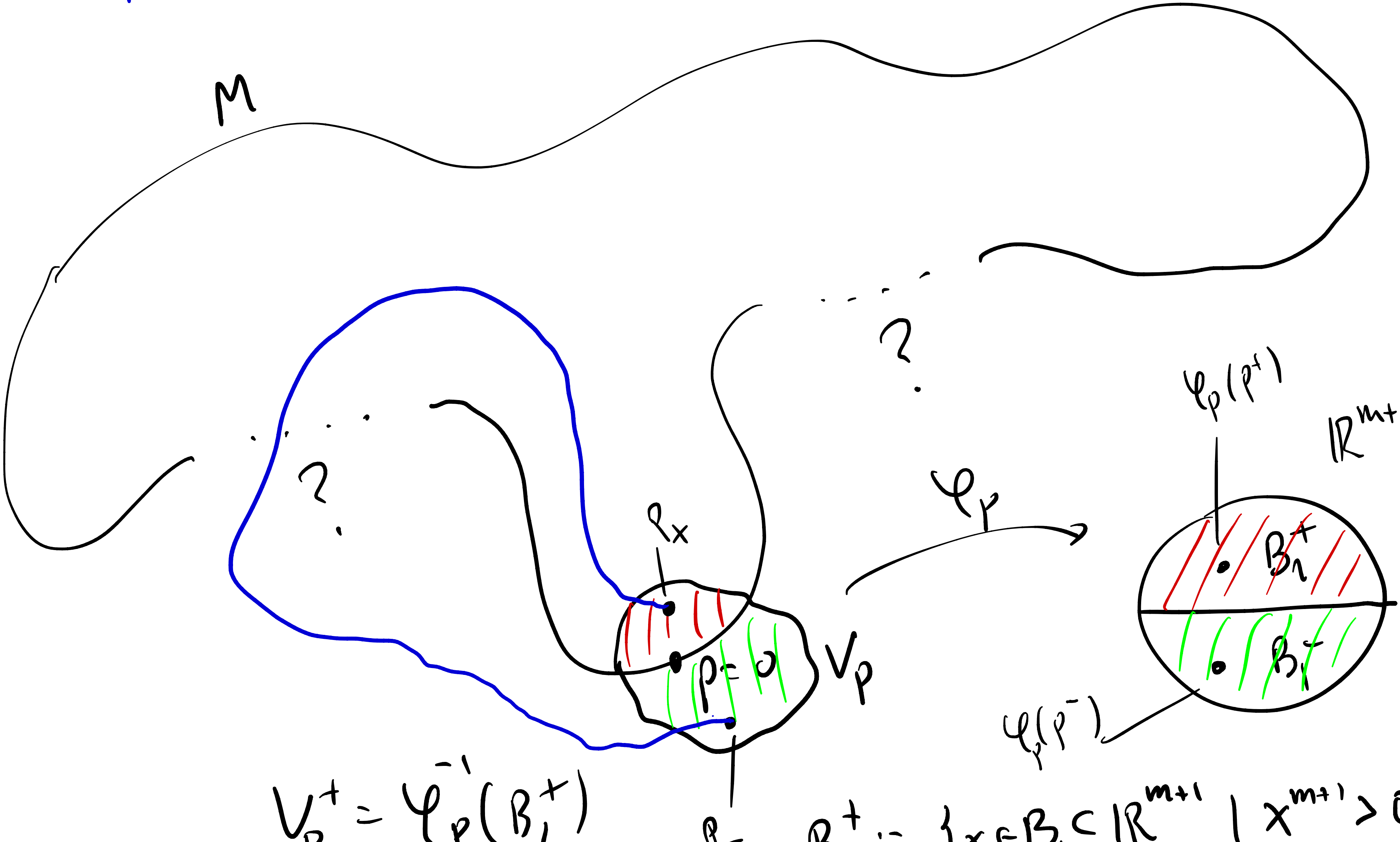
$$\tilde{\varphi}_p : \tilde{V}_p \longrightarrow U \qquad \tilde{\varphi}_p(p) = x$$

$$\varphi_p := \left.\frac{\tilde{\varphi}_p - x}{\rho}\right|_{\tilde{\varphi}^{-1}(B_\rho(x))} \qquad \rho > 0 \text{ is s.t. } B_\rho(x) \subset U$$



"Idea" of proof of step 1

We will suppose $\exists \gamma : [0,1] \longrightarrow \mathbb{R}^{m+1} \setminus M$ $\quad \gamma(0) = p^+$, $\gamma(1) = p^-$

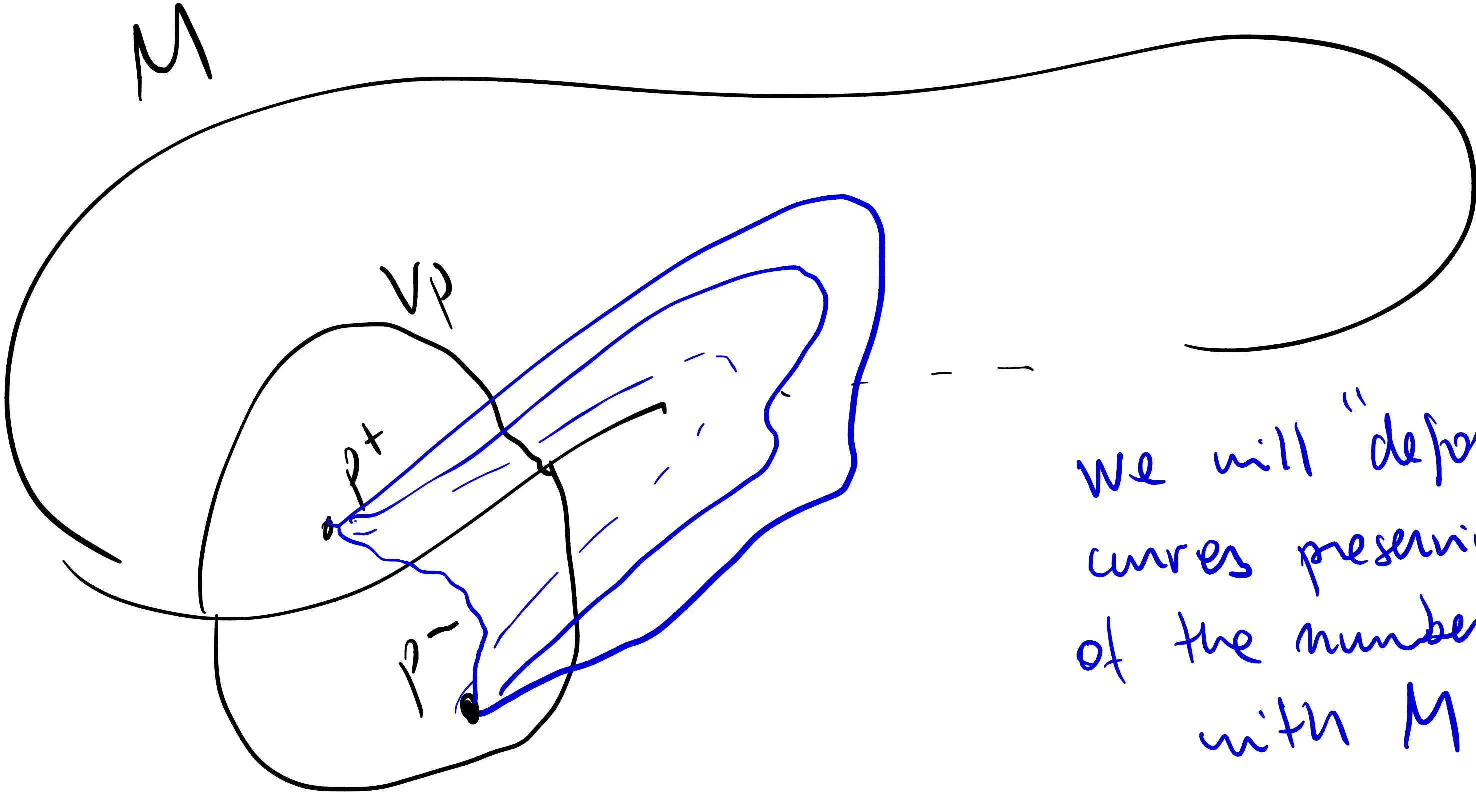


$$V_p^+ = \varphi_p^{-1}(B_1^+)$$

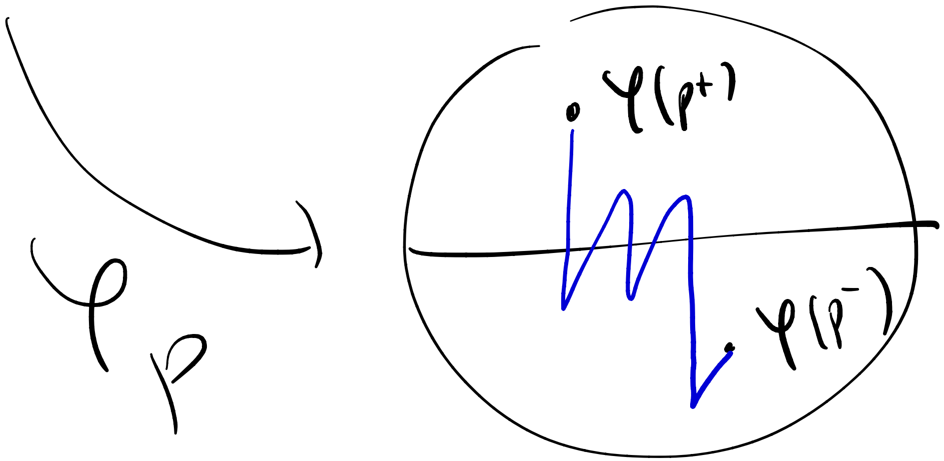
$$V_p^- = \varphi_p^{-1}(B_1^-)$$

$$B_1^+ := \{ x \in B_1 \subset \mathbb{R}^{m+1} \mid x^{m+1} > 0 \}$$

$$B_1^- := \{ \quad < 0 \}$$



we will "deform" the curves preserving the parity of the number of intersections with M



"discretized" curve

Given $0 = t_1 < t_2 < \dots < t_\ell = 1$

and an ℓ-tuple of pts in $\mathbb{R}^{m+1} \setminus M$ $\quad (q_1, \dots, q_\ell) = Q \in \mathbb{R}^{m+1} \setminus M$

such that $d(q_i, q_{i+1}) \leq \delta$ $\quad \gamma_Q : [0,1] \to \mathbb{R}^{m+1}$

define a piecewise smooth curve γ associated to Q as follows:

- If q_i and q_{i+1} are <u>both</u> at a dist $> \delta$ from M

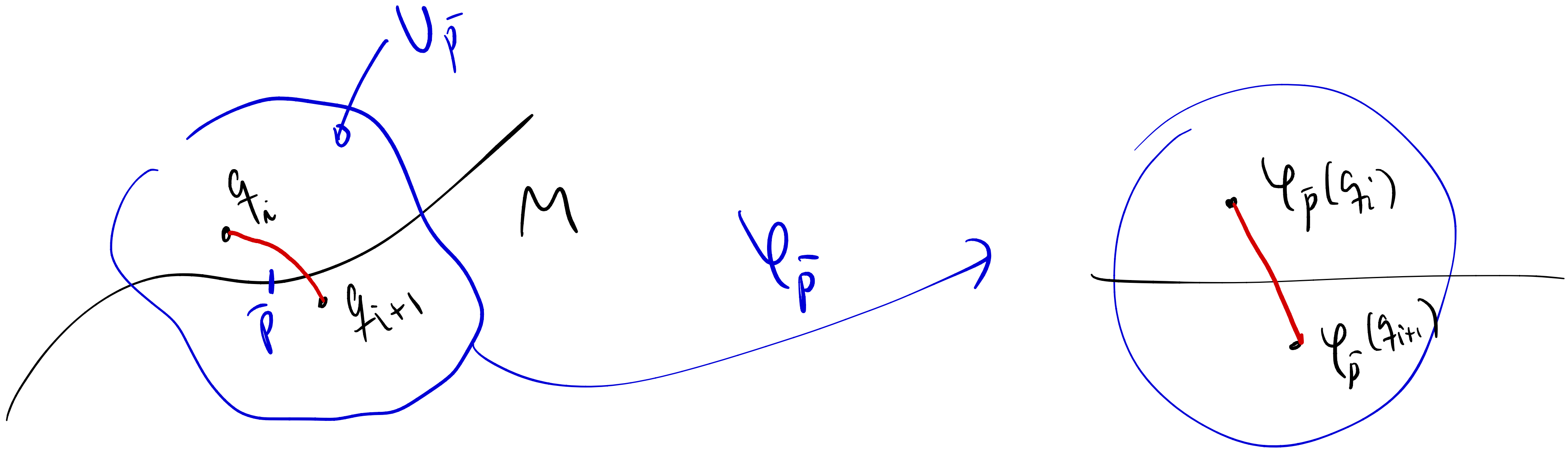
  define $$\gamma(t) = \frac{q_i(t_{i+1} - t) + q_{i+1}(t - t_i)}{t_{i+1} - t_i} \qquad t \in [t_i, t_{i+1}]$$

- If one of the 2 pts q_i or q_{i+1} is at dist $\leq \delta$ from M

  then we joint them by a segment but composed with some submanifold chart

Pick $\bar{p} \in M$ s.t. both q_i, q_{i+1} belong to $V_{\bar{p}}$

$$\gamma(t) = \varphi_{\bar{p}}^{-1}\left(\frac{\varphi_{\bar{p}}(q_i)(t_{i+1} - t) + \varphi_{\bar{p}}(q_{i+1})(t - t_i)}{t_{i+1} - t_i} \right)$$



How do we choose δ ?

For each $p \in M$ let $r_p > 0$ be st $B_{r_p}(p) \subset V_p$

Since $M \subset \bigcup_{p \in M} B_{\frac{r_p}{2}}(p)$, M compact

$\Longrightarrow$ $\exists$ finite subcover $\{p_1, \ldots, p_k\}$

Taking $\delta_0 := \frac{1}{2} \min\{r_{p_1}, \ldots, r_{p_k}\}$

Every $\delta \in (0, \delta_0]$ works!

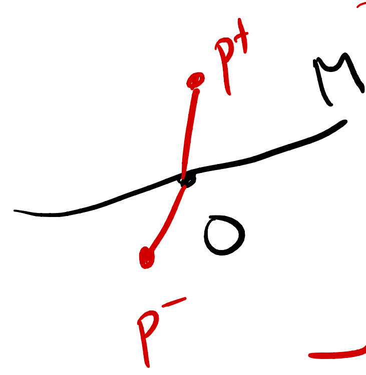
Now we can proceed with proof of step 1

Assume by contradiction that we have $p^+ \in V_p^+$, $p^- \in V_p^-$

$p = 0 \in M$, $\gamma : [0,1] \longrightarrow \mathbb{R}^{n+1} \setminus M$

$\gamma(0) = p^+$, $\gamma(1) = p^-$

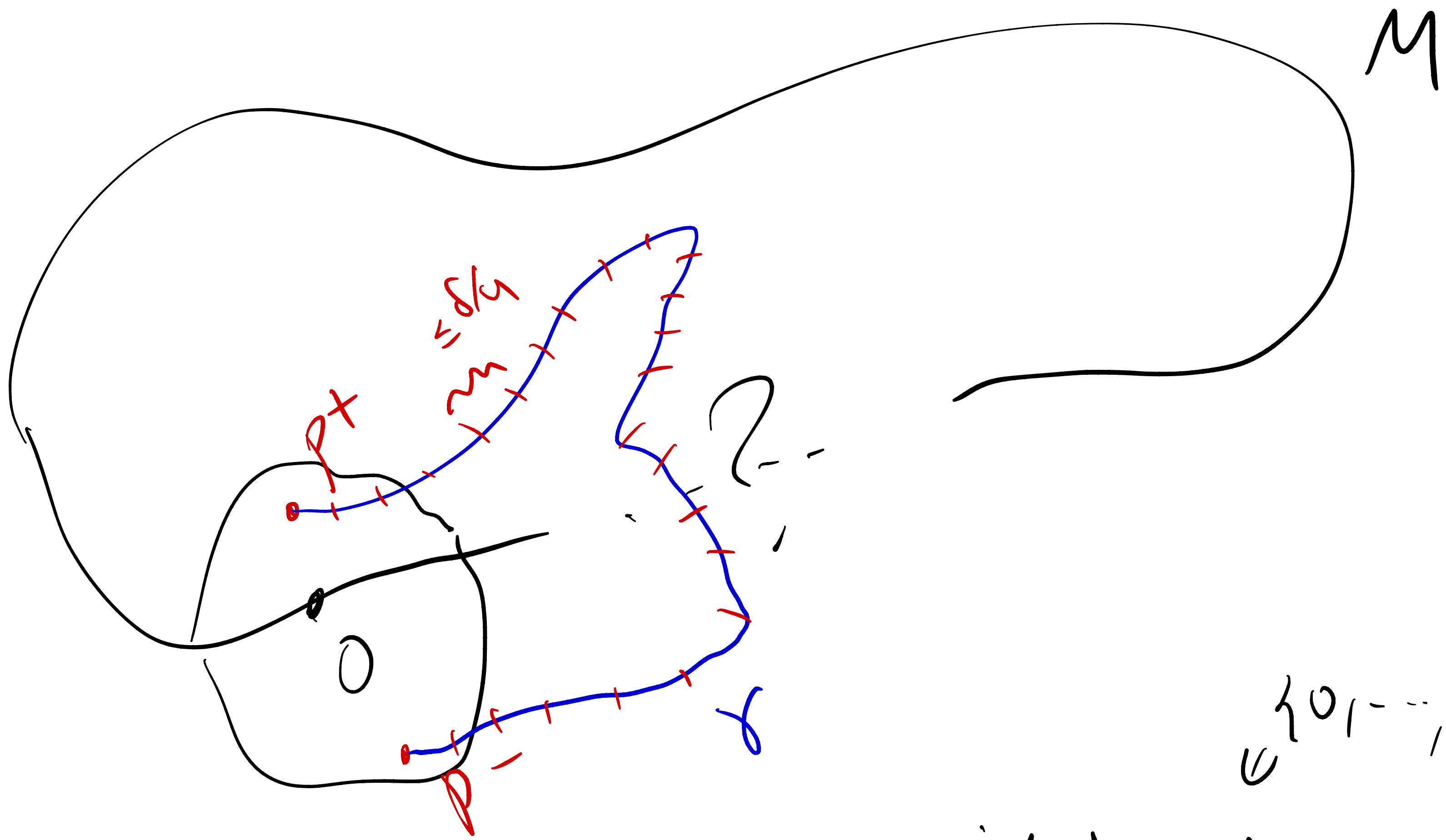
[Assume w.l.o.g. $[p^+, 0]$ segment joins p^+ and 0 intersects M only at 0, same with p^-]



Choose $\delta = \min(\delta_0, d(\gamma([0,1]), M))$ and ℓ large such that

$\exists$ $0 = t_1 < t_2 < \dots < t_\ell = 1$ with

$$d(\gamma(t_i), \gamma(t_{i+1})) \leq \frac{\delta}{4}$$



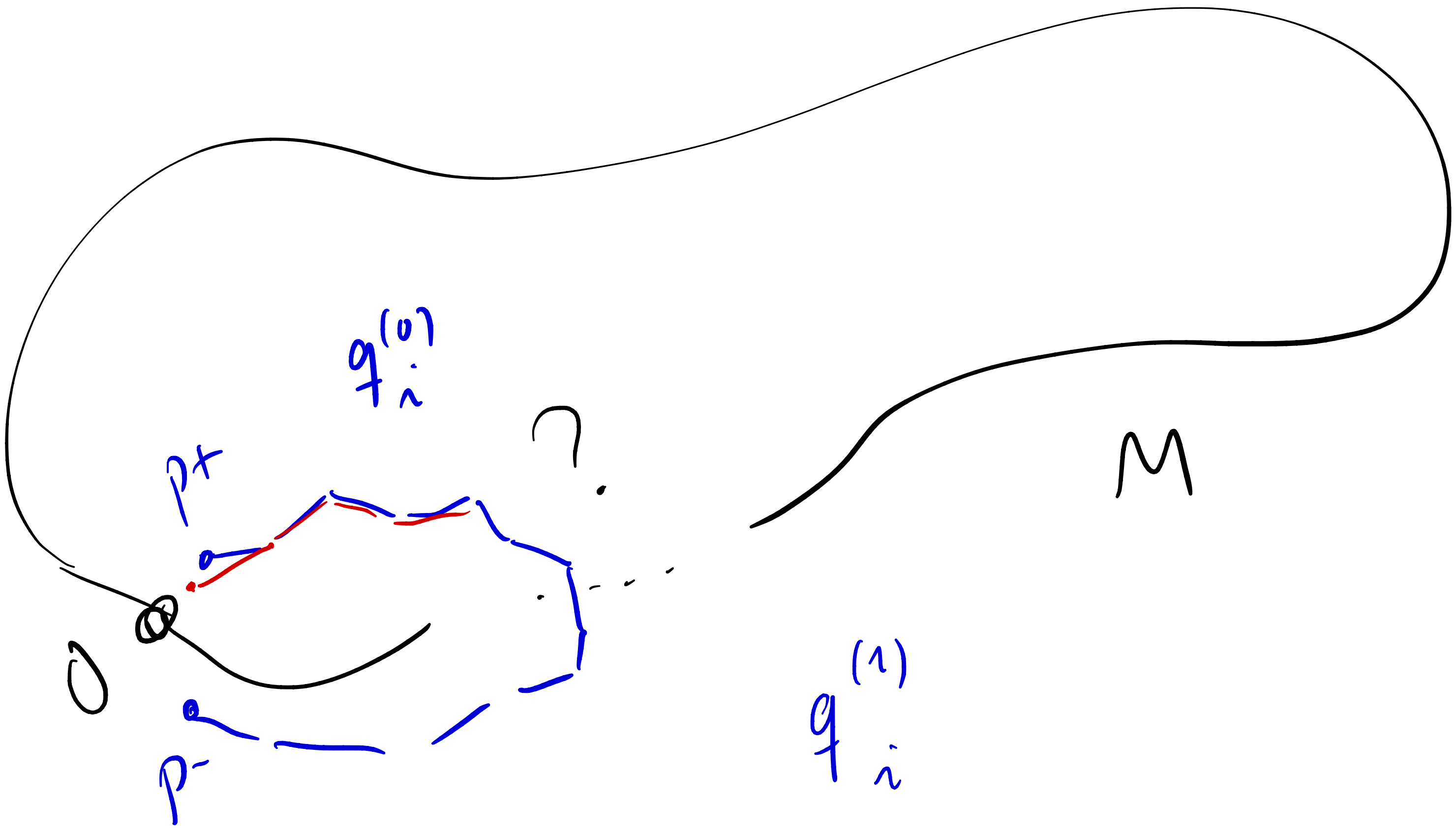
$\in \{0, \dots, \ell-1\}$

Define for every $j \geq 0$ $\quad j = \lfloor j/\ell \rfloor + j' \qquad 1 \le i \le \ell$

$$q_i^{(j)} = \begin{cases} \theta^{\lfloor j/\ell \rfloor} \gamma(t_i) + z_j & \text{if } i > j' \\ \theta^{\lfloor j/\ell \rfloor + 1} \gamma(t_i) + z_j & \text{if } i \le j' \end{cases}$$

$\theta < 1$ sufficiently close to 1

$z_j \in B_{\delta/4}(0)$ is chosen so that $q_i^{(j)} \notin M$



$\forall j \geq 0$ Given $Q^{(j)} = (q_1^{(1)}, \dots q_\ell^{(j)})$

I can conside $\gamma_{Q^{(j)}}(t)$

Observe:

1) $\gamma_{Q^{(0)}}([0,1]) \cap M = \emptyset$

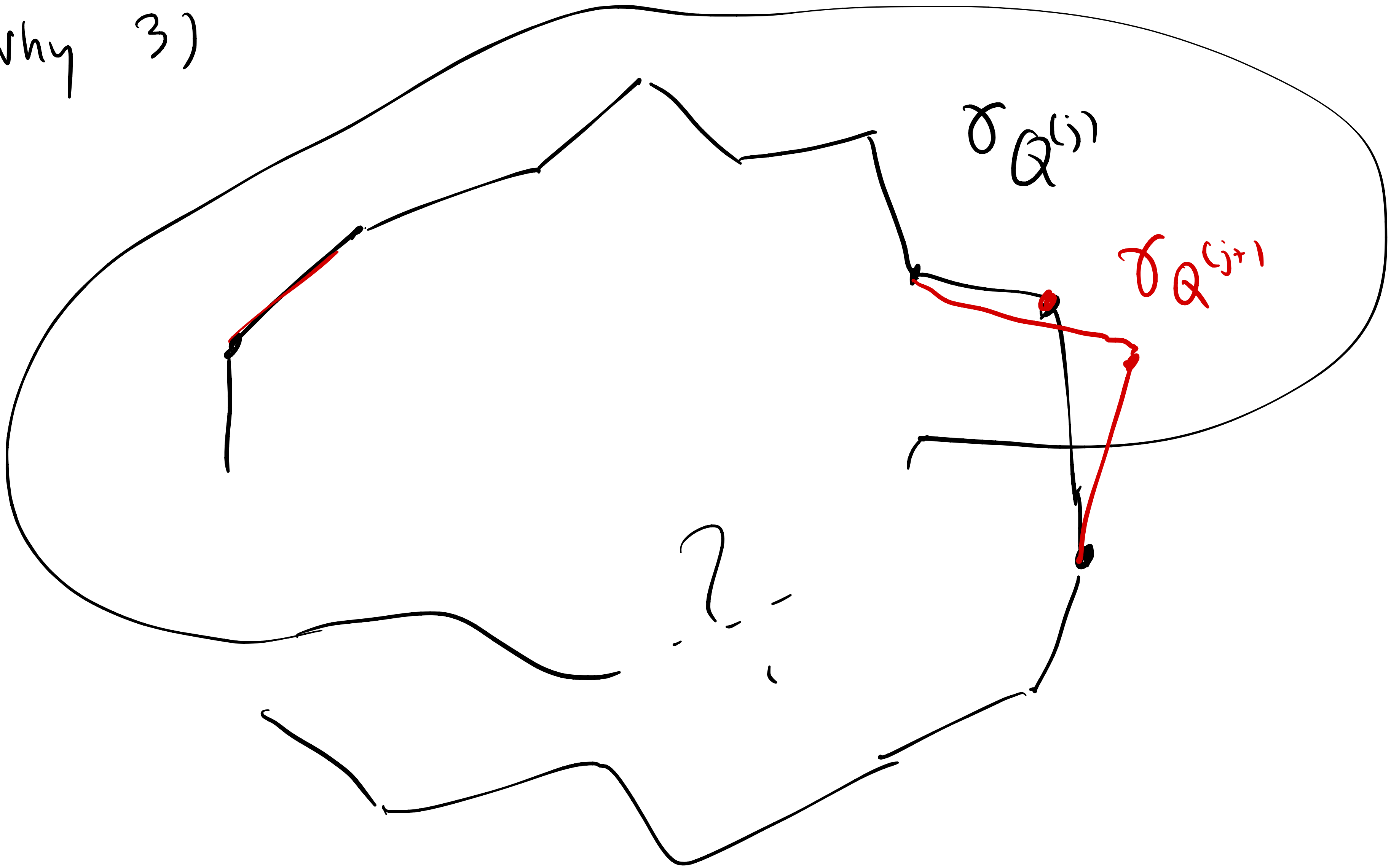
2) for j very large (since $\theta < 1$)

$\gamma_{Q^{(j)}}([0,1]) \subset V_p$ $\quad p = 0$

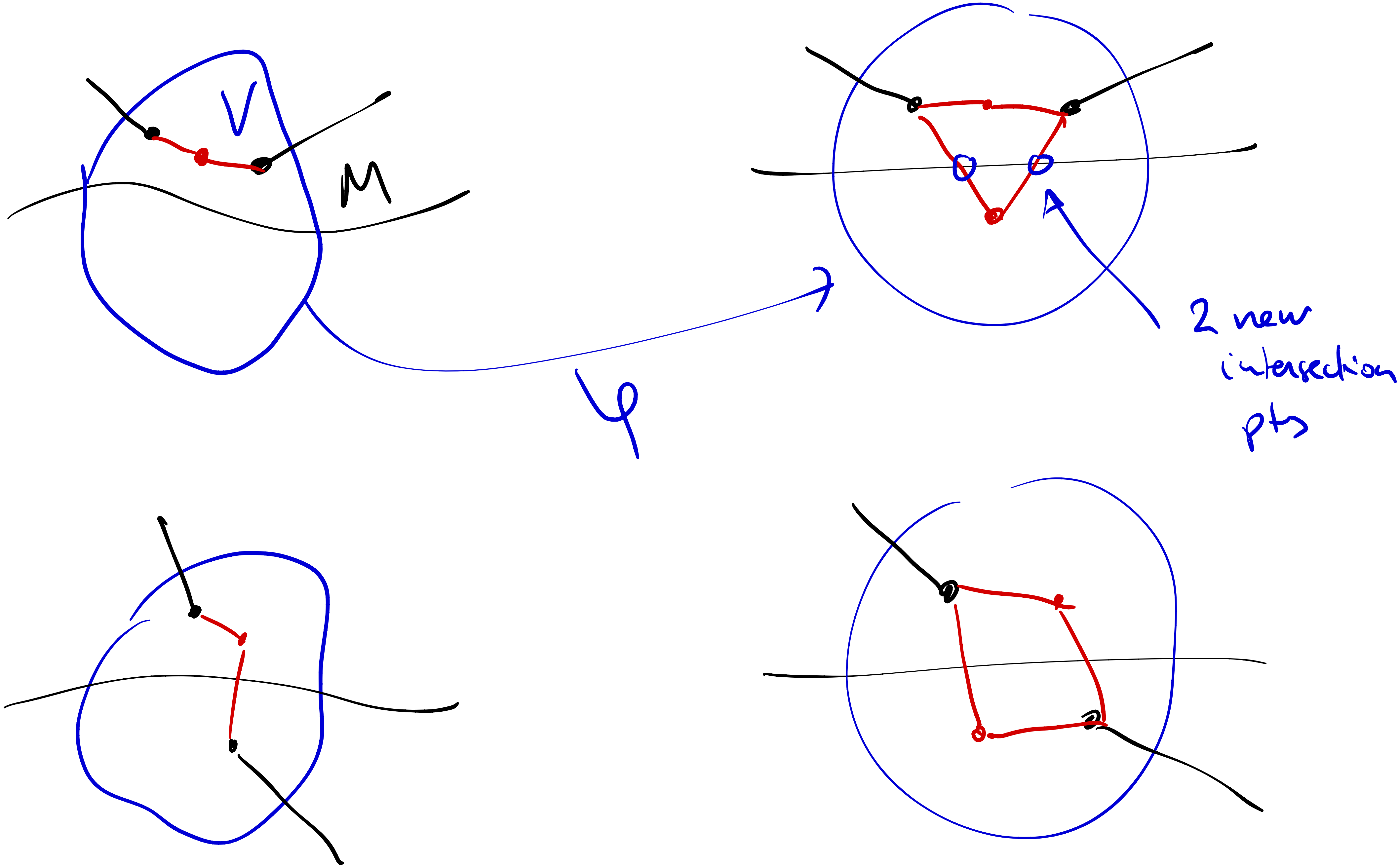
3) since from $\gamma_{Q^{(j)}}$ to $\gamma_{Q^{(j+1)}}$ I am updating only 1 pt, we have

$$\# \gamma_{Q^{(j)}}([0,1]) \cap M = \# \gamma_{Q^{(j+1)}}([0,1] \cap M) \mod 2$$

Why 3)



when the only point updated is close to M



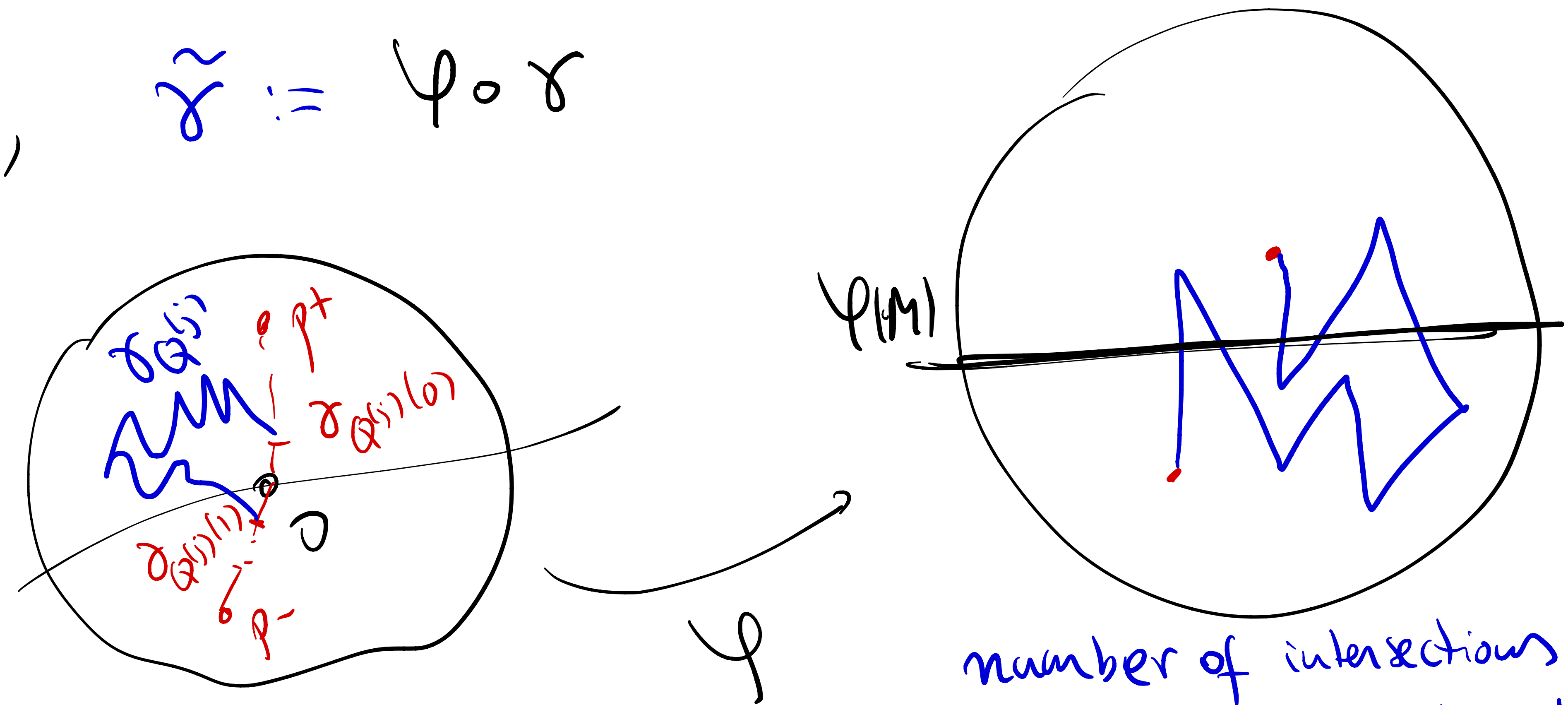
Final contradiction

by 2) in the jth step (j large)

$$\gamma_{Q^{(j)}}([0,1]) \subset V$$

domain of submanifold chart φ

so, $\tilde{\gamma} := \varphi \circ \gamma$



number of intersections with $\varphi(M)$ must be odd

<u>Step 2</u> By step 1 $\forall\, p, q \in M$ $\exists\, A_p, B_p$

<u>diffent conected comp</u> of $\mathbb{R}^{m+1} \setminus M$, sam with q : A_q, B_q

we would like to prove that either :

- $A_q = A_p$ and $B_q = B_p$
- $A_q = B_p$ and $B_q = A_p$

To do so, we use that M is connected (and smooth)

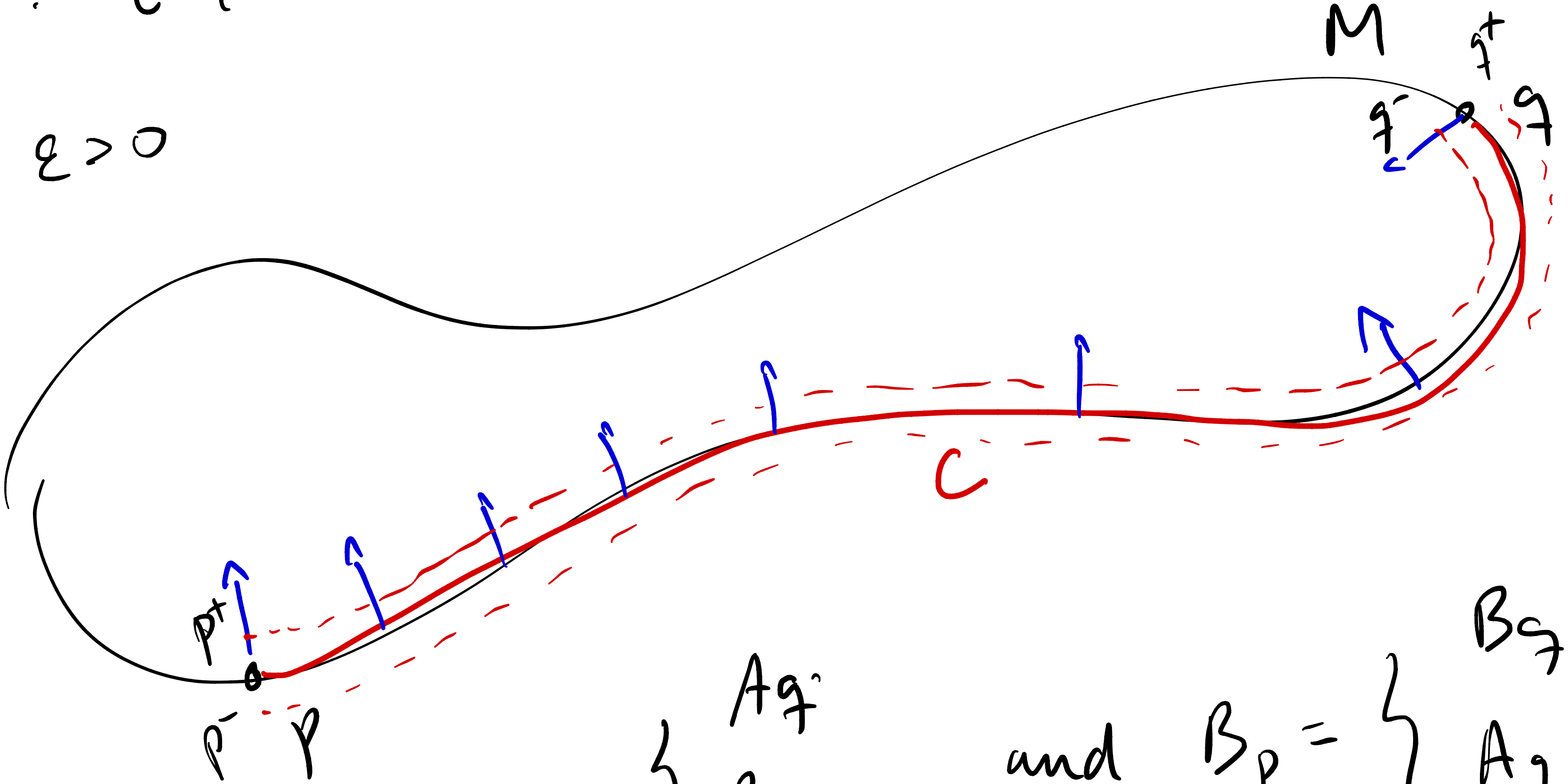
Choose $c : [0,1] \longrightarrow M$ s.t. $c(0) = p$, $c(1) = q$

Let $N : [0,1] \to S^{m+1} \subset \mathbb{R}^{m+1}$ be a continuous unit normal vector field along c ; that is $N(t) \in TM^{\perp}_{c(t)}$ $\forall\, t \in [0,1]$

Now consider $c^+, c^- : [0,1] \longrightarrow \mathbb{R}^{m+1} \setminus M$

$$c^\pm : t \longmapsto c(t) \pm \varepsilon N(t)$$

$\varepsilon > 0$



therefor $A_p = \begin{cases} A_{q^-} \\ B_q \end{cases}$ and $B_p = \begin{cases} B_q \\ A_q \end{cases}$

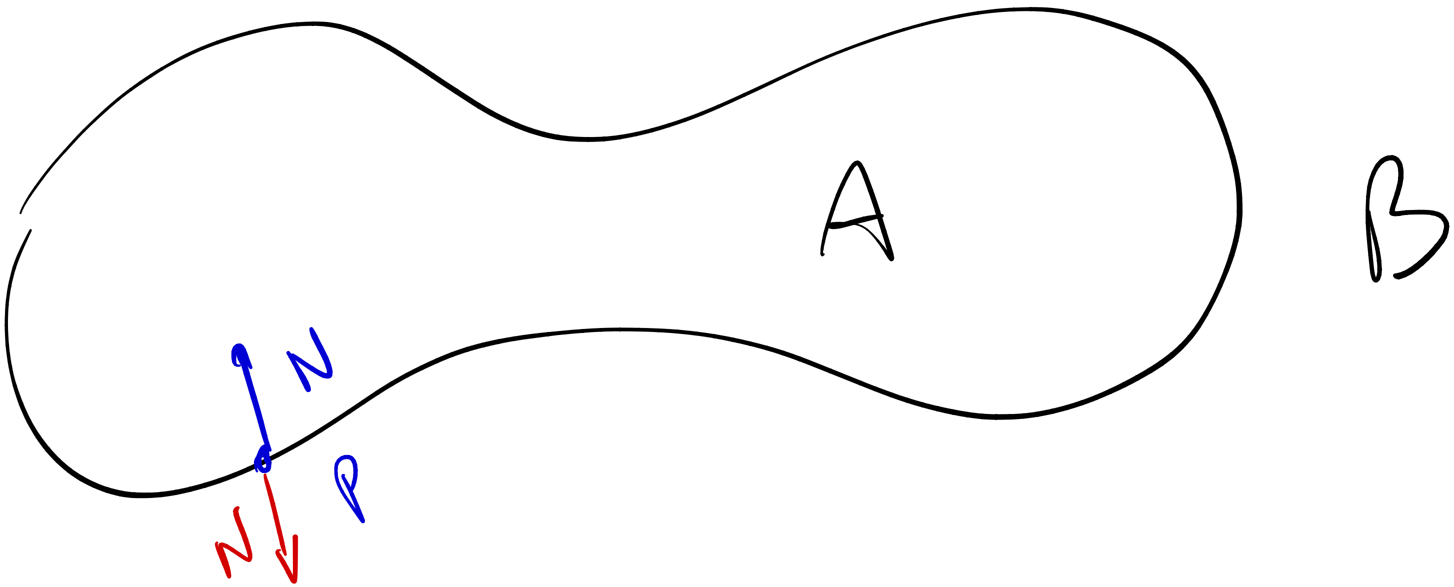
Therefore by steps 1 and 2 we showed that

$\mathbb{R}^{n+1} \setminus M$ consists of exactly 2 conn. components A, B

This allows us to define a continuous unit normal v.f

$N : M \to \mathbb{S}^{m+1}$ $\qquad N(p) \in TM_p^{\perp}$ $\qquad$ (<u>Gauss map</u>)

by the criterion $N(p)$ points towards A



Using Lemme 2.9, we deduce M is orientable

<u>Exercice</u> : prove that 1 of the con. comp is bdd
(and the other unbdd)

$\blacksquare$

PART 3

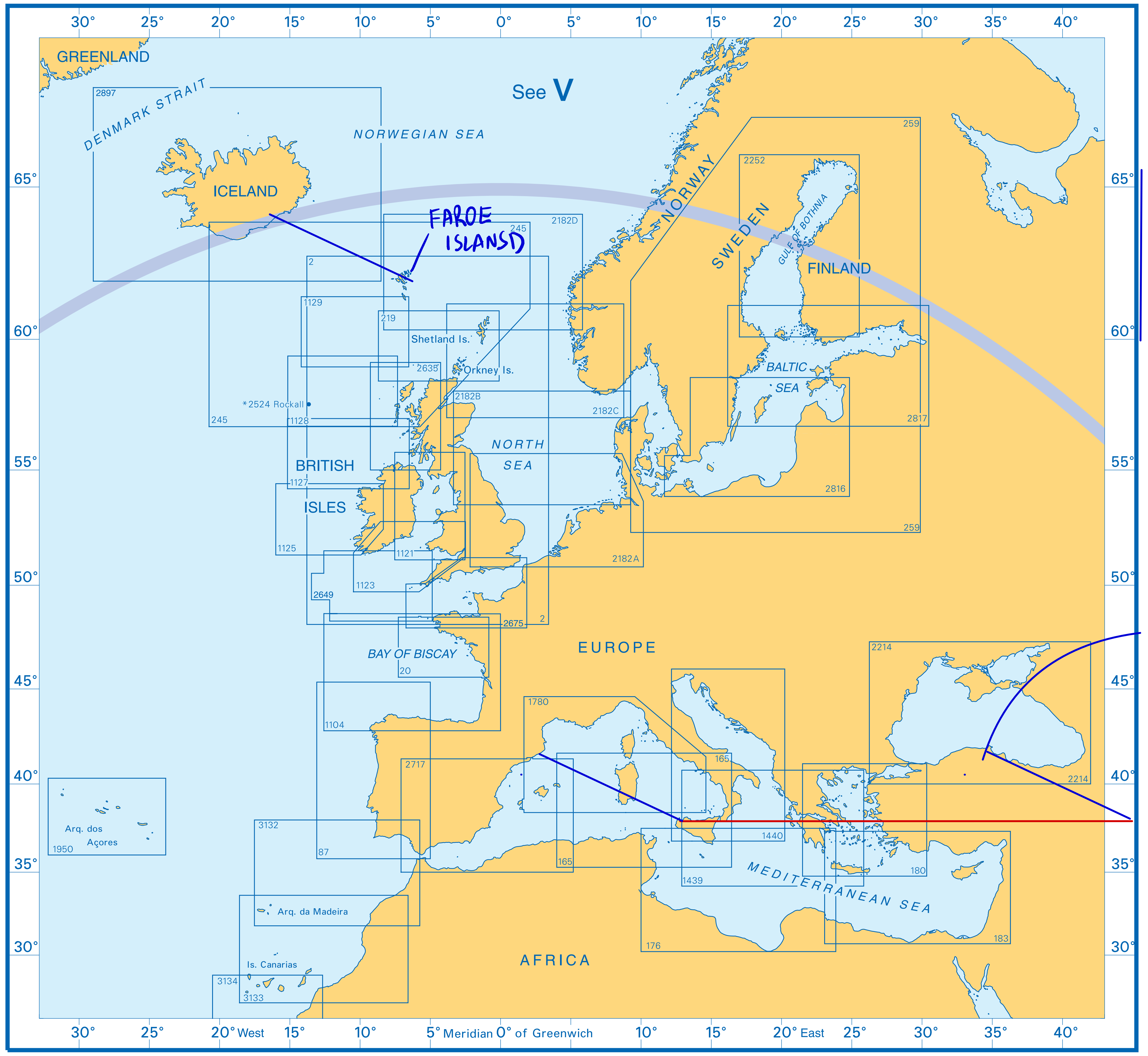
A2

North East Atlantic Ocean

European Waters

Mediterranean Sea -

Small Scale Charts



~ 10°

1° lat

||

60 miles

1 mile

||

1.85 Km

Chart No.		Title of Chart or Plan	Natural Scale 1:	Date of Publication	New Edition
2	⊙I	British Isles	1,500,000	Dec. 1983	July 2009
20	⊙	Île d'Ouessant to Pointe de la Coubre	500,000	Apr. 1990	-
87	⊙	Cabo Finisterre to the Strait of Gibraltar	1,000,000	Dec. 1972	Oct. 2010
165	⊙	Menorca to Sicilia including Malta	1,100,000	Sept. 1969	May 2007
176	⊙I	Cap Bon to Ra's At Tin	1,175,000	July 1988	May 2007
180	⊙	Aegean Sea	1,100,000	June 1973	Dec. 1988
183	⊙	Ra's At Tin to İskenderun	1,100,000	Oct. 1969	Mar. 1992
219	⊙I	Western Approaches to the Orkney and Shetland Islands	500,000	May 1983	Apr. 2009
245	⊙	Scotland to Iceland	1,250,000	July 1983	Apr. 2011
259	⊙I	Baltic Sea	1,500,000	July 1995	Oct. 2010
1104	⊙	Bay of Biscay	1,000,000	Nov. 1972	Mar. 2004
1121	⊙I	Irish Sea with Saint George's Channel and North Channel	500,000	Dec. 1980	Nov. 2000
1123	⊙	Western Approaches to Saint George's Channel and Bristol Channel	500,000	Dec. 1980	Dec. 2006
1125	⊙I	Western Approaches to Ireland	500,000	Sept. 1979	Jan. 1985
1127	⊙I	Outer Approaches to the North Channel	500,000	Dec. 1979	May 2007
1128	⊙	Banks West of the Hebrides	500,000	Sept. 1979	May 2007
1129	⊙	Banks North-west of the Hebrides	500,000	Mar. 1979	Aug. 2004
1439	⊙	Sicilia to Nisos Kriti	1,100,000	Nov. 1974	Dec. 2005
1440	⊙	Adriatic Sea	1,100,000	June 1969	May 2007
1780	⊙	Barcelona to Napoli including Islas Baleares, Corse, and Sardegna	1,100,000	Feb. 1972	Mar. 1993
1950	⊙	Arquipélago dos Açores	750,000	May 1994	Apr. 2011
2182A	⊙I	North Sea – Southern Sheet	750,000	Aug. 1973	July 2008
2182B	⊙I	North Sea – Central Sheet	750,000	Aug. 1973	May 2011
2182C	⊙I	North Sea – Northern Sheet	750,000	Dec. 1992	May 2011
2182D	⊙I	Norwegian Sea Føroyar to Bergen	750,000	Feb. 1980	May 2011

Chart No.		Title of Chart or Plan	Natural Scale 1:	Date of Publication	New Edition
2214	⊙	Black Sea including Marmara Denizi and Sea of Azov	1,200,000	Dec. 1995	-
2252	⊙	Gulf of Bothnia	750,000	Oct. 1971	Feb. 1997
2524	⊙	Islands off the North West Coast of Scotland	-	Sept. 1977	Sept. 2009
		Ⓐ Sula Sgeir	15,000		
		Ⓑ Rona	20,000		
		Ⓒ Sule Skerry	100,000		
		Ⓓ Rockall	50,000		
		Ⓔ Flannan Isles	15,000		
		Ⓕ Saint Kilda and Boreray	25,000		
2635	⊙I	Scotland – West Coast	500,000	Mar. 1980	May 2007
2649	⊙I	Western Approaches to the English Channel	500,000	Aug. 1978	Mar. 2003
2675	⊙I	English Channel	500,000	Aug. 1978	Mar. 2003
2717	⊙	Strait of Gibraltar to Barcelona and Alger including Islas Baleares	1,100,000	Jan. 1969	May 1995
2816	⊙	Baltic Sea – Southern Sheet	750,000	Jan. 1980	Oct. 2010
2817	⊙	Baltic Sea – Northern Sheet and Gulf of Finland	750,000	Mar. 1995	Oct. 2010
2897	⊙I	Iceland	1,000,000	May 2008	Apr. 2011
3132	⊙	Strait of Gibraltar to Arquipélago da Madeira	1,250,000	Sept. 1990	Oct. 2010
3133	⊙	Casablanca to Islas Canarias (including Arquipélago da Madeira)	1,250,000	Apr. 1992	Oct. 2006
		Ⓐ Ilhas Selvagens	100,000		
3134	⊙	Islas Canarias to Nouakchott	1,250,000	Dec. 1992	Oct. 2006

South West England and Brittany - Index B

Chart No.		Title of Chart or Plan	Natural Scale 1:	Date of Publication	New Edition
34	⊙	Isles of Scilly	25,000	Feb. 1972	June 2001
304	⊙I	Lorient and Approaches	10,000	Feb. 1997	Jan. 2002
		Ⓐ Le Blavet – Lanester to Hennebont	12,500		
442	⊙I	Lizard Point to Berry Head	150,000	Feb. 1978	Sept. 2002
777	⊙	Land's End to Falmouth	75,000	Feb. 1972	June 2001
883	⊙	Isles of Scilly. Saint Mary's and the Principal Off-Islands	12,500	Oct. 1983	June 2001
1076	⊙	Linney Head to Oxwich Point	75,000	July 1975	Nov. 2001
1121	⊙I	Irish Sea with Saint George's Channel and North Channel	500,000	Dec. 1980	Nov. 2000
1123	⊙	Western Approaches to Saint George's Channel and Bristol Channel	500,000	Dec. 1980	Dec. 2006
1148	⊙	Isles of Scilly to Land's End	75,000	Feb. 1972	May 2009
1149	⊙	Pendeen to Trevose Head	75,000	Feb. 1972	May 2009
1152	⊙	Bristol Channel – Nash Point to Sand Point	50,000	Sept. 1993	Sept. 2004
1156	⊙	Trevose Head to Hartland Point	75,000	Aug. 1973	Nov. 2010
1160	⊙	Harbours in Somerset and North Devon	-	Sept. 1974	June 2002
		Ⓐ Lynmouth	20,000		
		Ⓑ Porlock	20,000		
		Ⓒ Minehead	20,000		
		Ⓓ Watchet	20,000		
		Ⓔ Lundy	25,000		
		Ⓕ Barnstaple and Bideford	25,000		
		Ⓖ Ilfracombe	12,500		
1161	⊙I	Swansea Bay	25,000	Mar. 1994	Nov. 2001
		River Neath	12,500		
1164	⊙	Hartland Point to Ilfracombe including Lundy	75,000	Sept. 1974	Mar. 2010
1165	⊙	Bristol Channel – Worms Head to Watchet	75,000	Aug. 1976	Jan. 2006
1166	⊙	River Severn – Avonmouth to Sharpness and Hock Cliff	-	Oct. 1975	Aug. 2010
		Ⓐ Avonmouth to Severn Bridge	25,000		
		Ⓑ Severn Bridge to Sharpness	25,000		
		Ⓒ Sharpness to Hock Cliff	25,000		
		Ⓓ Sharpness Docks	10,000		
1167	⊙	Burry Inlet	25,000	Mar. 1981	Nov. 2001
1169	⊙	Approaches to Porthcawl	25,000	Aug. 1976	Mar. 2011
1176	⊙I	Severn Estuary – Steep Holm to Avonmouth	40,000	Sept. 1975	Feb. 2004
		Newport	20,000		
1178	⊙I	Approaches to the Bristol Channel	200,000	Dec. 1979	May 2009
1179	⊙I	Bristol Channel	150,000	Mar. 1979	Nov. 2004
1182	⊙I	Barry and Cardiff Roads with Approaches	25,000	May 1974	Jan. 2011
		Ⓐ Barry Docks	12,500		
		Ⓑ Cardiff Docks	15,000		
1410	⊙I	Saint George's Channel	200,000	Oct. 1980	Jan. 2002
1432	⊙I	Le Four to Ile Vierge	25,000	Dec. 1989	May 2008
		Aber Wrac'h	15,000		
1478	⊙	Saint Govan's Head to Saint David's Head	75,000	Aug. 1975	Nov. 2001
1482	⊙	Plans on the South and West Coasts of Dyfed	-	May 1975	Nov. 2001
		Ⓐ Ramsey Sound with the Bishops and Clerks	25,000		
		Ⓑ Jack Sound	12,500		
		Ⓒ Tenby and Saundersfoot with Approaches	25,000		
1613	⊙	Eddystone Rocks to Berry Head	75,000	Feb. 1972	Dec. 2005
		Eddystone Rocks	7,500		
1859	⊙I	Port of Bristol	-	July 1990	Dec. 2010
		Ⓐ King Road	10,000		
		Ⓑ River Avon	10,000		
		Ⓒ City Docks	5,000		
		Ⓓ City Docks to Saint Anne's Bridge	25,000		
1973	⊙	Cardigan Bay – Southern Part	75,000	Jan. 1975	Jan. 2002
2025	⊙	Portsall to Anse de Kernic	50,000	Apr. 2009	-
2026	⊙	Anse de Kernic to Ile Grande	50,000	Apr. 2009	-
2027	⊙	Ile Grande to Ile de Bréhat	48,700	May 2009	-
2028	⊙	Ile de Bréhat to Plateau des Roches Douvres	48,600	May 2009	-
2029	⊙	Ile de Bréhat to Cap Fréhel	48,800	May 2009	-
		Port Saint-Brieuc le Légué	10,000		
2049	⊙I	Old Head of Kinsale to Tuskar Rock	150,000	Mar. 1979	Oct. 2010
2348	⊙	Raz de Sein	20,000	Dec. 2000	Aug. 2008
		Ⓐ Port de Sein	10,000		
2349	⊙	Baie de Douarnenez	30,000	Oct. 2000	Jan. 2007
		Ⓐ Port de Morgat	10,000		
		Ⓑ Port de Douarnenez	10,000		
2350	⊙	Pointe de Saint-Mathieu to Chaussée de Sein	50,000	Oct. 2000	Jan. 2010
2356	⊙	Goulet de Brest to Portsall including Ile d'Ouessant	49,100	Apr. 2005	-
2357	⊙	Baie de Quiberon	20,000	Oct. 1997	May 2007
2371	⊙	Golfe du Morbihan	20,000	Apr. 2010	-
		Ⓐ Continuation of Rivière D'Auray Port De Saint Goustan	20,000		
		Ⓑ Continuation of Port de Vannes	20,000		
		Ⓒ Continuation of Rivière de Noyalo	20,000		
2454	⊙I	Start Point to the Needles including Off Casquets TSS	150,000	May 1977	Feb. 2008
2565	⊙I	St. Agnes Head to Dodman Point including the Isles of Scilly	150,000	Aug. 1978	May 2009
2643	⊙I	Île Vierge to Pointe de Penmarc'h	150,000	July 2011	-

Key to symbols

⊙ ARCS version. See page 14.

I International Chart. See pages 176 - 177.

3. Intrinsec geometry of surfaces

3.1 Def   1st fundamental form g of a submfld $M \subset \mathbb{R}^n$ assigns to every $p \in M$ the inner product

$$g_p(X,Y) := \langle X, Y \rangle_{\mathbb{R}^n} \qquad X, Y \in TM_p \subset \mathbb{R}^n$$

The 1st FF g of an immersion $f: U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$, assigns to every $x \in U$ the inner prod

$$g_x(\xi, \eta) := \langle df_x(\xi), df_x(\eta) \rangle \qquad \xi, \eta \in TU_x = \mathbb{R}^m$$

Remarks: 1. f immersion $\Rightarrow$ df_x injective $\Rightarrow$ g_x pos. def. symmetric, bilinear form

2. The matrix $(g_{ij}(x))$ of g_x w.r.t the standard basis $e_1, \dots, e_m$ of $\mathbb{R}^m$ is given by

$$g_{ij}(x) = g_x(e_i, e_j) = \langle df_x(e_i), df_x(e_j) \rangle = \left\langle \frac{\partial f}{\partial x^i}(x), \frac{\partial f}{\partial x^j}(x) \right\rangle$$

or: $$(g_{ij}) = J_f^T \cdot J_f \qquad J_f = \left(\frac{\partial f^i}{\partial x^j} \right)$$

(U, g) is a "model" of $f(U) \subset M$ ("like a nautical chart")

in which all intrinsic quantities of $f(U)$ can be computed

Examples

1. Norms of (tangent) vectors and angles: $f(x)=p$, $X = df_x(\xi)$
$Y = df_x(\eta)$

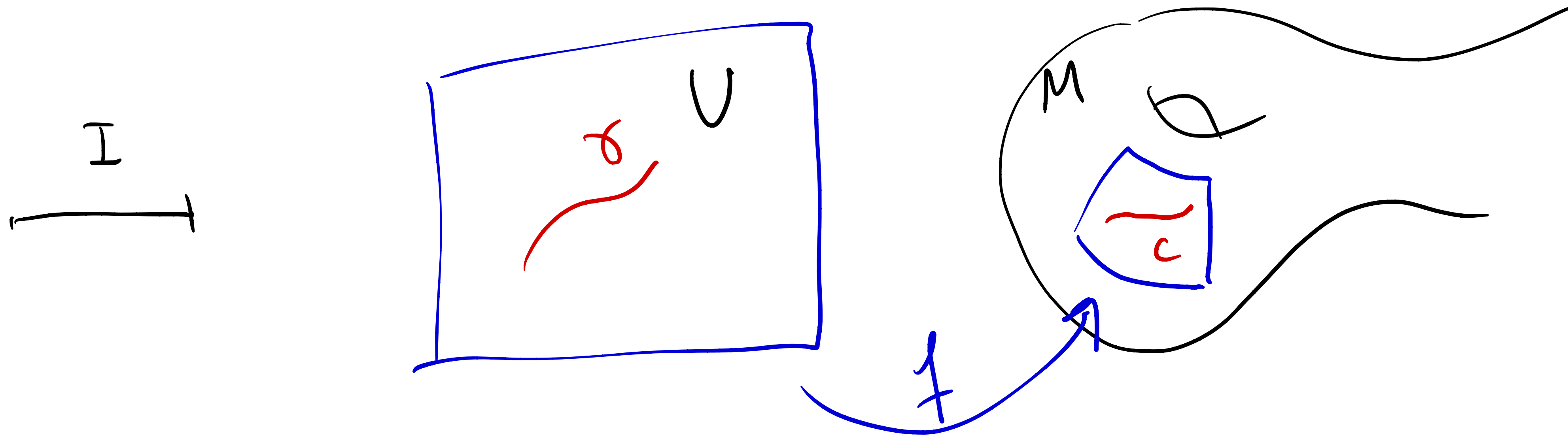
$$|X| = \sqrt{\langle X, X\rangle} = \sqrt{g_p(X,X)} = \sqrt{g_x(\xi,\xi)} =: |\xi|_{g_x}$$

$$\cos\angle(X,Y) = \frac{g_p(X,Y)}{|X|\,|Y|} = \frac{g_x(\xi,\eta)}{|\xi|_{g_x}\,|\eta|_{g_x}}$$

2. Length of curve $c : I \to f(U) \subset M$

$$\gamma := f^{-1}\circ c : I \longrightarrow U \qquad (c = f\circ\gamma)$$

$$L(c) = \int_I |c'(t)|\,dt = \int |\gamma'(t)|_{g_{\gamma(t)}}\,dt$$



3. m-dim. area of Borel sets $B \subset f(U) \subset M$

$$A(B) = \int_{f^{-1}(B)} \sqrt{\det(g_{ij}(x))}\, dx \in [0, \infty]$$

The Gram determinant $\det(g_{ij}(x)) = \det(\langle f_i(x), f_j(x)\rangle)$ equals the square of the volume of the parallelepiped in TM_p $(p = f(x))$ spanned by the vectors $f_i(x) := \frac{\partial f}{\partial x^i}(x)$

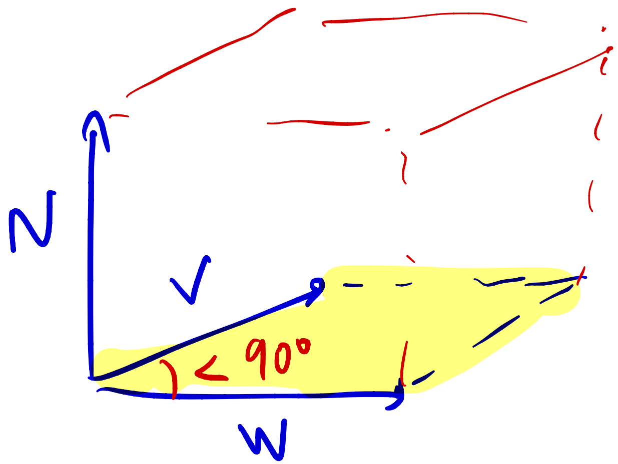
$i = 1, \dots, m$

Indeed, choose $f_{m+1}(x), \dots f_n(x)$ ONB of $TM_p^\perp$

$\boxed{\text{Volume spanned by } f_1, \dots f_m} = |\det(f_1, \dots, f_n)|$

$= \sqrt{\det\left((f_1, \dots f_n)^T \cdot (f_1, \dots, f_n)\right)} = \sqrt{\det\begin{pmatrix} g_{ij} & 0 \\ 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \end{pmatrix}}$

$= \sqrt{\det(g_{ij})}$



volume (▭) = heigth x area of base

Invariance under reparam. (of vol)

$$\begin{tikzcd} & B & \\ f^{-1}(B) \arrow[ur, "f"] \arrow[rr, "\psi"'] & & \tilde{f}^{-1}(B) \arrow[ul, "\tilde{f}"'] \end{tikzcd}$$

$B \subset f(U) \cap \tilde{f}(\tilde{U})$

$$df_x = d\tilde{f}_{\psi(x)} \circ d\psi_x$$

$$\boxed{g_{ij} = J_f^T \cdot J_f = (J_{\tilde{f}} J_\psi)^T (J_{\tilde{f}} J_\psi) = J_\psi^T (J_{\tilde{f}}^T J_{\tilde{f}}) J_\psi}$$

$$A(B) = \int_{f^{-1}(B)} \sqrt{\det(g_{ij}(x))}\, dx = \int_{f^{-1}(B) = \psi^{-1}(\tilde{f}^{-1}(B))} |\det J_\psi(x)| \sqrt{\det(\tilde{g}_{ij}(\psi(x)))}$$

$$\underset{\substack{\text{transl.}\\ \text{formula } \mathbb{R}^m}}{=} \int_{\tilde{f}^{-1}(B)} \sqrt{\det(\tilde{g}_{ij}(\tilde{x}))}\, d\tilde{x} = A(B)$$

To compute $A(K)$ for compact set $K \subset M$

Choose finitely many local param $f: U_\alpha \longrightarrow M$ and

Borel sets $B_\alpha \subset f_\alpha(U_\alpha)$ s.t. $K = \bigcup_\alpha B_\alpha$ disjoint union

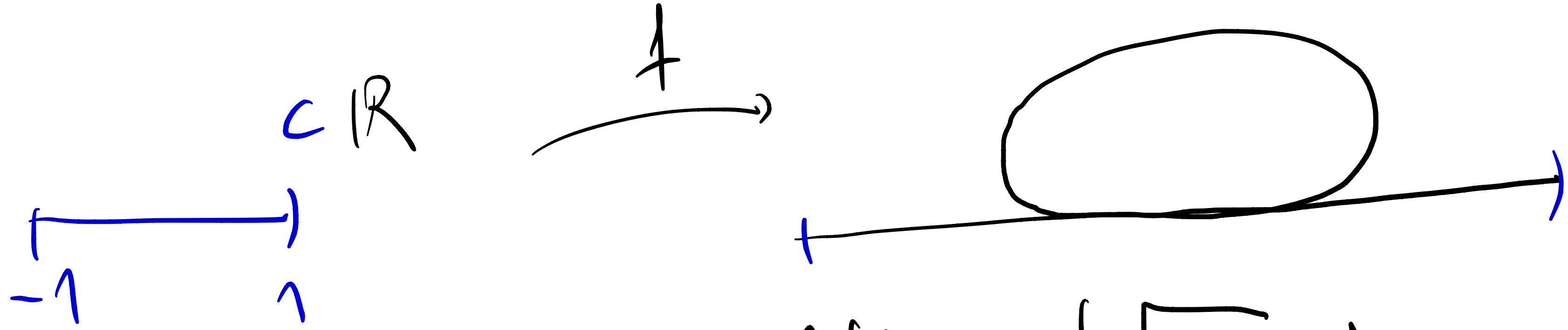
Then:

$$A(K) = \sum_\alpha A(B_\alpha) = \sum_\alpha \int_{f_\alpha^{-1}(B_\alpha)} \sqrt{\det(\underbrace{g_{ij}^\alpha}_{\text{1st FF of } f_\alpha\,!}(x))}\, dx$$

For a cont fcn $b: K \to \mathbb{R}$

$$\int_K b \, dA = \sum_\alpha \int_{f_\alpha^{-1}(B_\alpha)} b \circ f_\alpha(x) \underbrace{\sqrt{\det(g_{ij}^\alpha(x))} \, dx}_{dA \text{ "in coordinates"}}$$

Remark



$$A(f) = \int_U \sqrt{\det g_{ij}} \, dx$$

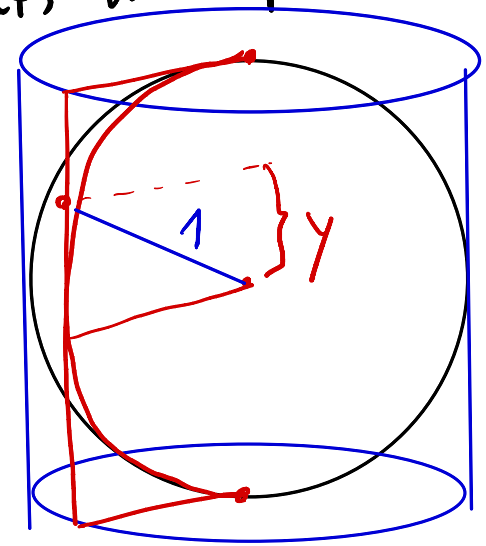
counts multiplicities

Examples

1. S^2 is cylindrical coordinates

$$f: \mathbb{R} \times (-1,1) \longrightarrow \mathbb{R}^3$$

$$f(x,y) = \left(\sqrt{1-y^2} \cos x, \sqrt{1-y^2} \sin x, y \right)$$



$$\frac{\partial f}{\partial x}(x,y) = \left(-\sqrt{}\,\sin x,\ \sqrt{}\,\cos x,\ 0\right)$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{-y}{\sqrt{}}\cos x,\ \frac{-y}{\sqrt{}}\sin x\ \ 1\right)$$

$$g_{ij}(x,y) = \begin{pmatrix} 1-y^2 & 0 \\ 0 & \frac{y^2}{1-y^2}+1 \end{pmatrix}$$

$$\frac{1}{1-y^2}$$

$\det(g_{ij}) = 1 \Rightarrow f$ is area preserving

$$A(S^2) = A\left(f|_{(0,2\pi)\times(-1,1)}\right)$$

$$= \int_{-1}^{1}\int_{0}^{2\pi}\sqrt{1}\,dx\,dy = 4\pi$$

2. Graphs $\quad U \subset \mathbb{R}^m, \quad f: U \longrightarrow \mathbb{R}^{m+1}$

$$f(x) = (x^1, \dots, x^m, h(x)) \qquad h: U \longrightarrow \mathbb{R} \quad C^\infty$$

f immersion, $f(U)$ submanifold

$$f_i = \frac{\partial f}{\partial x^i} = (0, \dots, 0, \overset{i}{\check{1}}, \dots 0, h_i), \quad h_i = \frac{\partial f}{\partial x^i}$$

$$g_{ij} = \delta_{ij} + h_i h_j$$

$$\nabla h = (h_1, \dots, h_m)$$

unit normal vector along f

$$\nu = \frac{1}{\sqrt{1 + |\nabla h|^2}} (-h_1, \dots, -h_m, 1)$$

$$
\sqrt{\det(g_{ij})} = \det(f_1, \dots, f_m, \nu)
$$

$$
= \frac{1}{\sqrt{1+|\nabla h|^2}} \det \begin{array}{c} \begin{array}{cccc|c} & & & & \nu \end{array} \\ \left(\begin{array}{ccc|c} 1 & & & -h_1 \\ & \ddots & & \vdots \\ & & 1 & -h_m \\ \hline h_1 & \cdots & h_m & 1 \end{array} \right) \end{array}
$$

$$
= \frac{1 + \sum_{i=1}^{m} h_i^2}{\sqrt{1+|\nabla h|^2}} = \sqrt{1+|\nabla h|^2}
$$

$$
A(f) = A(f(U)) = \int_U \sqrt{1+|\nabla h|^2}\, dx
$$

3.2 Def'n Two submflds $M \subset \mathbb{R}^n$, $\tilde{M} \subset \mathbb{R}^{\tilde{n}}$ with 1st FF g and $\tilde{g}$ (respectively) are called <u>isometric</u> if $\exists$ diffeomorphism $F : M \to \tilde{M}$

$$g_p(X,Y) = \tilde{g}_{F(p)}(dF_p(X), dF_p(Y)) \qquad (1)$$

$\forall p \in M$, $X, Y \in TM_p$

Two immersions $f : U \to \mathbb{R}^n$, $\tilde{f} : \tilde{U} \to \mathbb{R}^{\tilde{n}}$ are <u>isometric</u> if $\exists$ diffeo $\Psi : U \to \tilde{U}$ st.

$$g_x(\xi,\eta) = \underbrace{\tilde{g}_{\Psi(x)}(d\Psi_x(\xi), d\Psi_x(\eta))}_{(*)} \qquad (2)$$

$\forall x \in U$, $\xi, \eta \in \mathbb{R}^m$

Standard notation "pull-back"

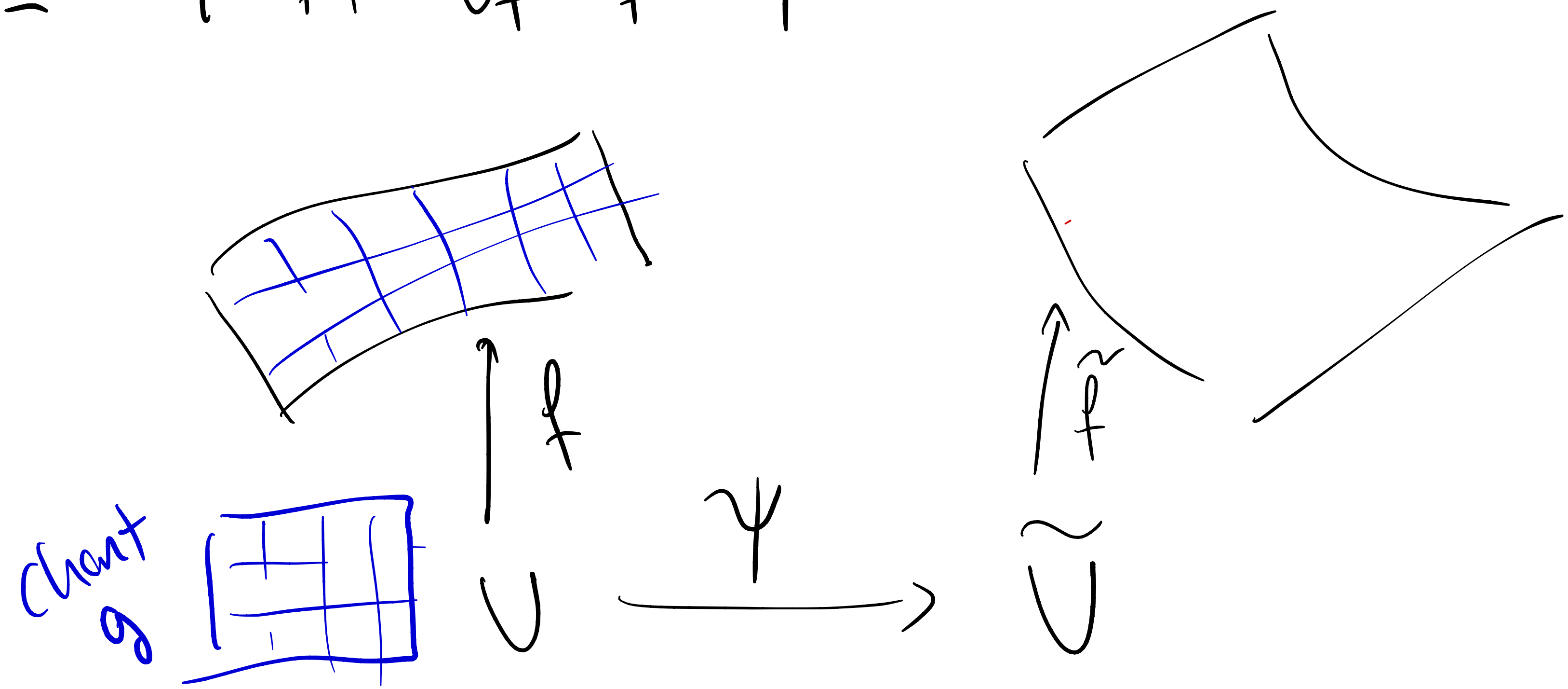
(1) $g = F^* \tilde{g}$

(2) $g = \Psi^* \tilde{g}$

$$(*) = \left\langle d\tilde{f}_{\Psi(x)}(d\Psi_x(\xi)),\ d\tilde{f}_{\Psi(x)}(d\Psi_x(\eta)) \right\rangle$$

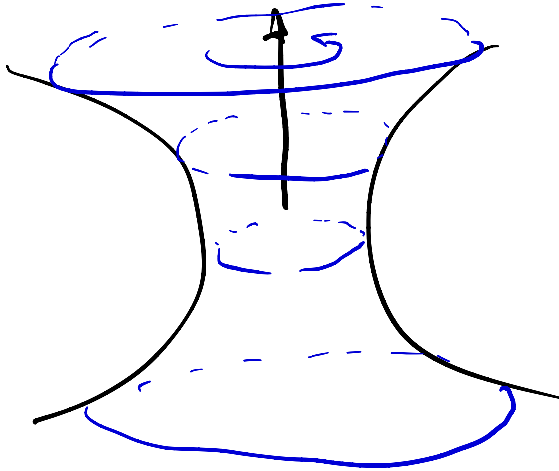
$$= \left\langle d(\tilde{f}\circ\Psi)_x(\xi),\ d(\tilde{f}\circ\Psi)_x(\eta) \right\rangle$$

$$= 1^{st}\ \text{FF of } \tilde{f}\circ\Psi$$



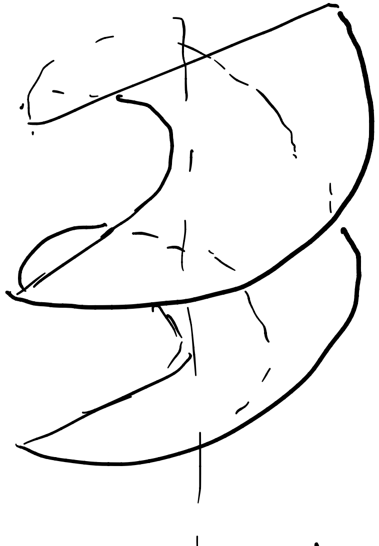
Example

catenoid



$$f(x,y) = (\cosh y \cos x,\ \cosh y \sin x,\ y)$$

helicoid



$$\tilde{f}(\tilde{x},\tilde{y}) = (\tilde{y}\cos\tilde{x},\ \tilde{y}\sin\tilde{x},\ \tilde{x})$$

these are isometric!

Define $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$

$$(\tilde{x},\tilde{y}) = \Psi(x,y) = (x, \sinh y)$$

$$\Rightarrow \tilde{f}\circ\Psi = (\sinh y \cos x,\ \sinh y \sin x,\ x)$$

exercise $(1\text{st FF of } f) = (1\text{st FF of } \tilde{f}\circ\Psi) = (\cosh y\ \delta_{ij})$

Covariant derivative

($f(U)$: my surface)

$f : U \to \mathbb{R}^n$ immersion , $U \subset \mathbb{R}^m$

The vector $\frac{\partial f}{\partial x^1}(x), \dots, \frac{\partial f}{\partial x^m}(x)$ form a basis of the tangent space $df_x(\mathbb{R}^m)$ of f at x

In general $\frac{\partial^2 f}{\partial x^i \partial x^j}(x) \notin df_x(\mathbb{R}^m)$

$\rightsquigarrow$ Consider tangential part

coefficients are called Christoffel symbols

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(x) \right)^T =: \sum_{k=1}^m \Gamma_{ij}^k(x) \frac{\partial f}{\partial x^k}(x)$$

orthogonal projection onto $df_x(\mathbb{R}^m)$

For i, j fixed

3.3 Lemma

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{m} g^{kl}\underbrace{\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)}_{*}$$

where (g^{kl}) is the inverse matrix to (g_{ij})

<u>Observation</u> In particular Γ_{ij}^{k} are computable in terms of g ⇝ these are intrinsic!

<u>proof</u> Notation $f_i = \frac{\partial f}{\partial x^i}$, $f_{ij} = \frac{\partial f}{\partial x^i \partial x^j}$, etc.

$$* \quad \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} = \frac{\partial}{\partial x^i}\langle f_j, f_l\rangle + \frac{\partial}{\partial x^j}\langle f_i, f_l\rangle - \frac{\partial}{\partial x^l}\langle f_i, f_j\rangle$$

$$= \langle f_i, f_{lj} - f_{jl}\rangle + \langle f_j, f_{li} - f_{il}\rangle + \langle f_l, f_{ji} + f_{ij}\rangle$$

$$= 2\langle f_\ell , f_{ij} \rangle = 2\langle f_\ell , f_{ij}{}^T \rangle$$

since f_ℓ is tangent

def'n Γ_{ij}^k

$$= 2\left\langle f_\ell , \sum_{k=1}^m \Gamma_{ij}^k f_k \right\rangle$$

$$= 2\sum_{k=1}^m \Gamma_{ij}^k \langle f_\ell , f_k \rangle = 2\sum_{k=1}^m \Gamma_{ij}^k g_{\ell k}$$

$$\Longrightarrow \sum_{k=1}^m \Gamma_{ij}^k g_{\ell k} = \frac{1}{2}\left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$$

$\times g^{\ell p}$ and sum over ℓ

(use $\sum_\ell g^{\ell p} g_{\ell k} = \delta^p_k$)

<u>3.4 def'n</u> $M \subset \mathbb{R}^n$ m-dim submfld, $c: I \to M$ curve

$X: I \longrightarrow \mathbb{R}^n$ C^1 tangent v.f. of M along c

(i.e. $X(t) \in TM_{c(t)}$ $\forall t \in I$)

<u>covariant derivative</u> $\frac{D}{dt} X : I \longrightarrow \mathbb{R}^n$

$$\frac{D}{dt} X(t) := \left(\dot{X}(t)\right)^T \in TM_{c(t)}$$

X is <u>parallel along</u> c if $\frac{D}{dt} X(t) \equiv 0$ $\forall t \in I$

$\left(\Leftrightarrow \dot{X}(t) \in TM_{c(t)}^{\perp}\right)$

3.5. Thm M, c as above, $X, Y : I \longrightarrow \mathbb{R}^n$ C^1 tangent vf of M along c

Then:

(1) $\frac{D}{dt}(X+Y) = \frac{D}{dt}X + \frac{D}{dt}Y$

(2) $\frac{D}{dt}(\lambda X) = \left(\frac{d}{dt}\lambda\right) \cdot X + \lambda \frac{D}{dt} X$ $\quad (\lambda : I \longrightarrow \mathbb{R}, C^1)$

(3) $\frac{d}{dt} g(X, Y) = g\left(\frac{D}{dt}X, Y\right) + g\left(X, \frac{D}{dt}Y\right)$

(4) If $f : U \to f(U) \subset M$ is local param.

$\gamma : I \to U$ $\quad c = f \circ \gamma$ $\quad \xi = (\xi^1, \dots \xi^m)$

$\xi : I \longrightarrow \mathbb{R}^m \; (= TU_\bullet)$

$$X(t) = df_{\gamma(t)}(\xi(t)) = \sum_{i=1}^m \xi^i(t) \frac{\partial f}{\partial x^i}(\gamma(t))$$

Then

$$\frac{D}{dt}X = \sum_{k=1}^{m}\left[\dot{\xi}^k + \sum_{i,j=1}^{m} \xi^i \dot{\gamma}^j (\Gamma^k_{ij}\circ\gamma)\right]\frac{\partial f}{\partial x^k}\circ\gamma$$

<u>proof</u> (1), (2) obvious from defn

$\frac{d}{dt}$

(3) $$\langle X, Y\rangle^{\cdot} = \langle \dot{X}, Y\rangle + \langle X, \dot{Y}\rangle$$
$$= \langle \dot{X}^T, Y\rangle + \langle X, \dot{Y}^T\rangle$$

(4) $\left(f_i := \frac{\partial f}{\partial x^i},\ f_{ij} := \frac{\partial^2 f}{\partial x^i \partial x^j}\right)$ $\boxed{X = \sum_{i=1}^{m} \xi^i (f_i\circ\gamma)}$

$$\left(\left(\xi^i (f_i\circ\gamma)\right)^{\cdot}\right)^T = \left(\dot{\xi}^i (f_i\circ\gamma)\right)^T + \xi^i \underbrace{\sum_{j=1}^{m} (f_{ij}\circ\gamma)^T \dot{\gamma}^j}_{\overset{\shortparallel}{(f_i\circ\gamma)^{\cdot}}}$$

take tangential parts!

with chain rule

Use defin Γ_{ij}^k:

$$(f_{ij}\circ\gamma)^T = \sum_{k=1}^{m} \Gamma_{ij}^k\, f_k\circ\gamma$$

$$\left(\left(\xi^i (f_i\circ\gamma)\right)^{\cdot}\right)^T = \dot{\xi}^i (f_i\circ\gamma) + \xi^i \sum_{j,k=1}^{m} \Gamma_{ij}^k (f_k\circ\gamma)\,\dot{\gamma}^j$$

sum in the variable i $\blacksquare$

Remarks

1. (4) shows that $\frac{D}{dt}X$ is intrinsic
2. If X, Y are parallel along c then $g_{c(t)}(X(t), Y(t))$ is constant

$$\frac{d}{dt} g(X,Y) \underset{(3)}{=} g\Big(\underbrace{\frac{D}{dt}X}_{=0}, Y\Big) + g\Big(X, \underbrace{\frac{D}{dt}Y}_{0}\Big) = 0$$