For a chart (φ, U) of M around p,

canonical derivations $\left.\frac{\partial}{\partial \varphi^i}\right|_p$ at p $\quad i=1,\dots,m$

defined by

$$\left.\frac{\partial}{\partial \varphi^i}\right|_p (f) := \frac{\partial f}{\partial \varphi^i}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

8.10 Prop The set of all derivations at $p \in M$ (C^∞) forms an m-dim vector space.
If φ is a chart around p, then

$$\left.\frac{\partial}{\partial \varphi^1}\right|_p, \dots, \left.\frac{\partial}{\partial \varphi^m}\right|_p \text{ is a basis,}$$

and any derivation X at p is of the form

$$X = \sum_{j=1}^{m} X(\varphi^j) \frac{\partial}{\partial \varphi^j}\Big|_p \qquad (*)$$

<u>proof</u> • $\frac{\partial}{\partial \varphi^j}\Big|_p$ are lin. indep.

$$\sum_{j=1}^{m} \lambda^j \frac{\partial}{\partial \varphi^j}\Big|_p = 0 \quad \underset{\substack{\text{apply to} \\ f = \varphi^i}}{\Longrightarrow} \quad 0 = \sum_{j=1}^{m} \lambda^j \frac{\partial \varphi^i}{\partial \varphi^j}(p) = \sum_{j=1}^{m} \lambda^j \delta^i_j = \lambda^i$$

• Show $(*)$. Assume w.log $\varphi(p) = 0$

Then, in nbhd of p, any f has a representation

$$f(q) = f(p) + \sum_{i=1}^{m} f_i(q)\, \varphi^i(q) \qquad \underline{\text{where}} \qquad f_i \in C^\infty(M)_p$$

Indeed $h := f \circ \varphi^{-1}$, $\varphi(q) =: x$

$$h(x) - h(0) = \left[h(tx)\right]_{t=0}^{t=1} = \int_0^1 \frac{d}{dt}\left[h(tx)\right] dt$$

$$= \int_0^1 \frac{\partial h}{\partial x^i}(tx) \frac{d(tx^i)}{dt} = \sum_i x^i \underbrace{\int_0^1 \frac{\partial h}{\partial x^i}(tx)}_{h_i(x)}$$

Take $\underline{f_i := h_i \circ \varphi}$.

Notice $q = p$ $x = 0$ $h_i(0) = f_i(p) = \frac{\partial h}{\partial x^i}(0) = \frac{\partial f}{\partial \varphi^i}(p)$

Now $\left.\frac{\partial}{\partial \varphi^j}\right|_p (f) = \left.\frac{\partial}{\partial \varphi^j}\right|_p \left(\sum_i f_i \varphi^i \right)$

$$= \sum_i \left.\frac{\partial}{\partial \varphi^j}\right|_p f_i \underbrace{\varphi^i(p)}_{0} + f_i(p) \delta_j^i$$

$$\Rightarrow \quad \frac{\partial}{\partial \varphi^j}\Big|_p f = f_i(p) = \frac{\partial f}{\partial \varphi^j}(p)$$

$$X(f) = X\Big(\sum_j f_j \varphi^j\Big) = \sum_j X(f_j)\underbrace{\varphi^j(p)}_{0} + f_j(p) X(\varphi^j)$$

$$= \sum_j X(\varphi^j) \frac{\partial f}{\partial \varphi^j}(p)$$

$$= \sum_j X(\varphi^j) \frac{\partial}{\partial \varphi^j}\Big|_p (f)$$

<u>Identification</u> tangent space $\longleftrightarrow$ derivations

M is C^∞ mfld, we identify $X \in TM_p$ (Def 8.4)

with the derivation

$$X(f) \overset{(*)}{:=} df_p(X) \in T\mathbb{R}_{f(p)} \cong \mathbb{R}$$

for $X = [\varphi, \xi]_p$

$$df_p(X) = df_p([\varphi,\xi]_p) \overset{\text{def}}{=} \left[\mathrm{id}_{\mathbb{R}},\ d(f\circ\varphi^{-1})_{\varphi(p)}(\xi)\right]_{f(p)}$$

$$= d(f\circ\varphi^{-1})_{\varphi(p)}(\xi) = \sum_j \frac{\partial(f\circ\varphi^{-1})}{\partial x^j}(\varphi(p))\,\xi^j$$

$$= \underbrace{\sum_j \xi^j \cdot \frac{\partial}{\partial \varphi^j}\Big|_p}_{X}(f)$$

For $F : M^m \longrightarrow N^n$ C^∞, $X \in TM_p$,

$f \in C^\infty(N)$, we have

$$dF_p(X)(f) \overset{(*)}{=} df_{F(p)}(dF_p(X))$$

$$= d(f \circ F)_p(X) \overset{(*)}{=} X(f \circ F)$$

$c : I \to M$ C^∞ $\quad c'(t) \in TM_{c(t)}$ as derivation

$$c'(t)(f) \underset{(*)}{=} df_{c(t)}(c'(t)) = (f \circ c)'(t)$$

Differential forms (and Stoke's thm)

$$\int_a^b f' = f(b) - f(a) \quad + \quad \text{terminology}$$

$\Lambda^s(\mathbb{R}^{n*}) :=$ vector space of alternating s-linear maps

$$\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{s} \longrightarrow \mathbb{R}$$

$$f(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(s)}) = \operatorname{sgn}(\sigma)\, f(\xi_1, \ldots, \xi_s)$$

In particular $\Lambda^0(\mathbb{R}^{n*}) = \mathbb{R}$, $\Lambda^s(\mathbb{R}^{n*}) = 0$, $s \geq n+1$

Exterior product

$s \geq 0$

$t \geq 0$

$\alpha \in \Lambda^s(\mathbb{R}^{n*})$, $\beta \in \Lambda^t(\mathbb{R}^{n*})$

$\alpha \wedge \beta \in \Lambda^{s+t}(\mathbb{R}^{n*})$

$(\alpha \wedge \beta)(\xi_1, \ldots, \xi_{s+t}) :=$

$$:= \sum_{(s,t)\text{-shuffles}} \operatorname{sgn}(\sigma)\, \alpha(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(s)})\, \beta(\xi_{\sigma(s+1)}, \ldots, \xi_{\sigma(s+t)})$$

$\{\sigma \in S_{s+t} \mid \sigma(1) < \sigma(2) < \ldots \sigma(s),$

$\sigma(s+1) < \sigma(\sigma+2) < \ldots < \sigma(s+t)\}$

Properties

- $\wedge$ bilinear
- $a \in \Lambda^0(\mathbb{R}^{n*}) \cong \mathbb{R}$ $\quad a \wedge \alpha = a\alpha$ $\left(\forall \alpha \in \Lambda^s(\mathbb{R}^{n*})\right)$
- $\alpha \wedge \beta = (-1)^{st} \beta \wedge \alpha$
- $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

$e_1, \dots, e_n$ denote canonical basis of $\mathbb{R}^n$

$e^1, \dots, e^n$ dual basis, i.e $e^i(e_j) = \delta^i_j$

Every $\alpha \in \Lambda^s(\mathbb{R}^{n*})$ has the representation

$\longrightarrow$ By def'n $e^i \in \mathbb{R}^{n*} = \Lambda^1(\mathbb{R}^{n*})$

$$\alpha \overset{(*)}{=} \sum_{1 \le i_1 < i_2 < \cdots < i_s \le n} \alpha_{i_1, \ldots, i_s} \, e^{i_1} \wedge \cdots \wedge e^{i_s}$$

$$\alpha(e_{i_1}, \ldots, e_{i_s}) \qquad i_1 < i_2 < \cdots i_s$$

$$\boxed{e^{i_1} \wedge \cdots \wedge e^{i_s}(\xi_1, \ldots, \xi_s) = \det \underbrace{\left(e^{i_\alpha}(\xi_\beta)\right)}_{1 \le \alpha, \beta \le s}}$$

<u>Def'n</u> A <u>differential form</u> ω <u>of degree</u> $s \geq 0$ on $U \subset \mathbb{R}^n$ open is a map $U \longrightarrow \Lambda^s(\mathbb{R}^{n*})$ such that given any s-tuple of vectors

$$(\xi_1, \ldots \xi_s) \in (\mathbb{R}^n)^s$$

$x \longmapsto \omega_x(\xi_1, \dots, \xi_s)$ is smooth (C^∞)

As a conseq. if $\xi_1, \dots \xi_s \in C^\infty(U, \mathbb{R}^n)$

$$\omega(\xi_1, \dots, \xi_s) : U \longrightarrow \mathbb{R}$$

$$\shortparallel$$

$$\omega_x(\xi_1(x), \dots, \xi_s(x))$$

Differential d of scalar fc'n as a diff'l form of deg 1

If $f : U \longrightarrow \mathbb{R}$ smoth

"of deg 1"

$df :=$ 1-diff form

$\xi \in \mathbb{R}^n$ $\qquad (df)_x(\xi) = \underbrace{df_x}_{\text{usual differential!}}(\xi)$

1-form

$f = x^i : \mathbb{R}^n \longrightarrow \mathbb{R}$

$dx^i = e^i \iff dx^i(e_j) = \delta^i_j$

→ From now on representations of s-forms will be of the type $\sin(x^1x^2)\, dx^1 \wedge dx^2$

Denote $\Omega^s(U)$, $U \subset \mathbb{R}^n$ open, the space of diff forms of deg s on U

For all $\omega \in \Omega^s(U)$ we have

$$\omega = \sum_{1 \le i_1 < \dots < i_s \le n} \overbrace{\omega_{i_1,\dots,i_s}}^{C^\infty(U)} \, dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

$$\overset{\text{"briefly"}}{=} \sum_I \omega_I \, dx^I \qquad \begin{cases} I = (i_1, \dots, i_s) \\ i_1 < \dots < i_s \end{cases}$$

As before

$$\omega_{i_1,\dots,i_s} = \omega(e_{i_1}, \dots, e_{i_s})$$

Theorem (exterior derivative) $\quad$ ($U \subset \mathbb{R}^n$ open)

$\exists$ unique sequence of linear operators

$$d : \Omega^s(U) \longrightarrow \Omega^{s+1}(U) \qquad s \geq 0$$

with the following properties:

(1) For $f \in \Omega^0(U) = C^\infty(U)$ $\quad$ df is the usual differential

(2) $d \circ d = 0$ $\quad$ $\left(\forall f \in \Omega^s(U),\ d(df) = 0 \in \Omega^{s+2} \right)$

(3) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^s \omega \wedge d\theta$

whenever $\omega \in \Omega^s(U)$, $\theta \in \Omega^t(U)$

(4) $d(\omega|_V) = d\omega|_V \quad \forall V \subset U$ open

---

proof

Uniqueness

$$\omega = \sum_J \underset{\in C^\infty(U) = \Omega^0(U)}{\omega_J}\, dx^J = \sum_J \omega_J \wedge dx^J$$

(1)+(3)

$$d\omega \overset{}{=} \sum_J d\omega_J \wedge dx^J + (-1)^0 \omega_J \wedge \underbrace{d(dx^J)}_{\overset{(2)}{\equiv} 0}$$

$$\overset{(*)}{=} \sum_{1 \leq i_1 < \dots < i_s \leq n} d\omega_{i_1,\dots,i_s} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

existence

Define d according to (*) (and (1))

check that it satisfies (1) — (4)

(2) $w = f\, dx^I$ on U

$$dw = \underbrace{df}\wedge dx^I = \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x^i} dx^i} \wedge dx^I$$

$$d(dw) = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x^j \partial x^i} \underbrace{dx^j \wedge dx^i}_{-dx^i \wedge dx^j} \wedge d^I = 0$$

Def'n $F : U \longrightarrow V$ C^∞ $\omega \in \Omega^s(V)$

$U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$

Pullback form $F^*\omega \in \Omega^s(U)$

$$(F^*\omega)_x(\xi_1, \dots, \xi_s) = \omega_{F(x)}(dF_x(\xi_1), \dots, dF_x(\xi_s))$$

In particular if $\omega = f \in C^\infty(V)$ 0-form $F^*\omega = \omega \circ F$

Proposition $F : U \longrightarrow V$ C^∞ (as above)

$\omega \in \Omega^s(V)$, $\theta \in \Omega^t(V)$. Then: $(a, b \in \mathbb{R})$

(0) $F^*(a\omega + b\theta) = aF^*\omega + bF^*\theta$ $\leftarrow$ if $s=t$!

(1) $F^*(\omega \wedge \theta) = F^*\omega \wedge F^*\theta$

(2) $F^*(d\omega) = d(F^*\omega)$

__proof__ Hint for (2). Prove it first for

$$w = f \in C^\infty(V) = \Omega^0(V) \iff \text{chain rule}$$

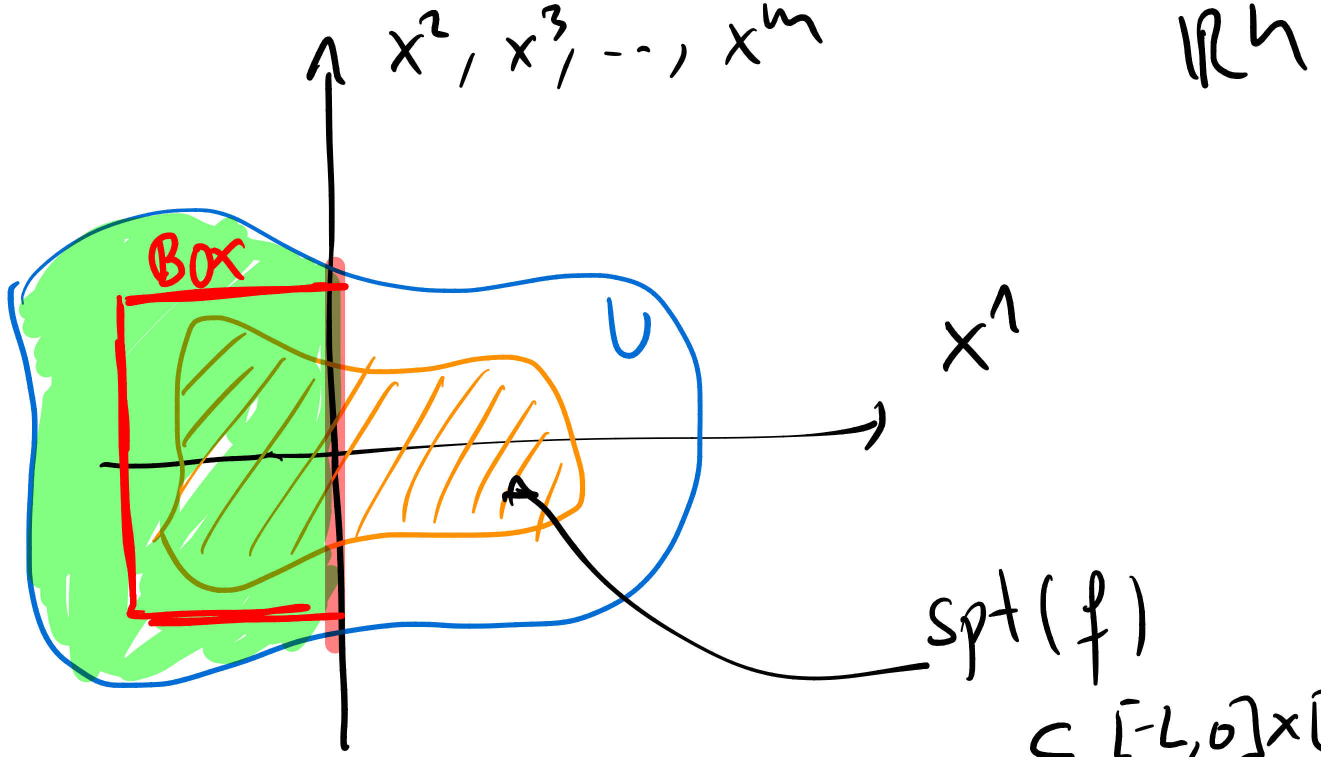
$$\big(d(f \circ F) = dF \circ dF \big)$$

use induction over $s \geq 0$ and Thm of exterior derivative.

Integration of forms and Stokes' thm

__Baby version of Stokes'__ $U \subset \mathbb{R}^m$ open $f \in C_c^\infty(U)$

$$(1) \quad \underbrace{\int_{U \cap \{x^1 < 0\}} \frac{\partial f}{\partial x^1}\, dx^1 \cdots dx^m}_{\text{LHS}} = \int_{U \cap \{x^1 = 0\}} f \, dx^2 \cdots dx^m$$



$\subset \underbrace{[-L,0]\times[-L,L]^{m-1}}_{\text{Box}}$

$2 \le i \le m$

(2) $$\int_{U\cap\{x^1<0\}} \frac{\partial f}{\partial x^i}\, dx^1 \cdots dx^m = 0$$

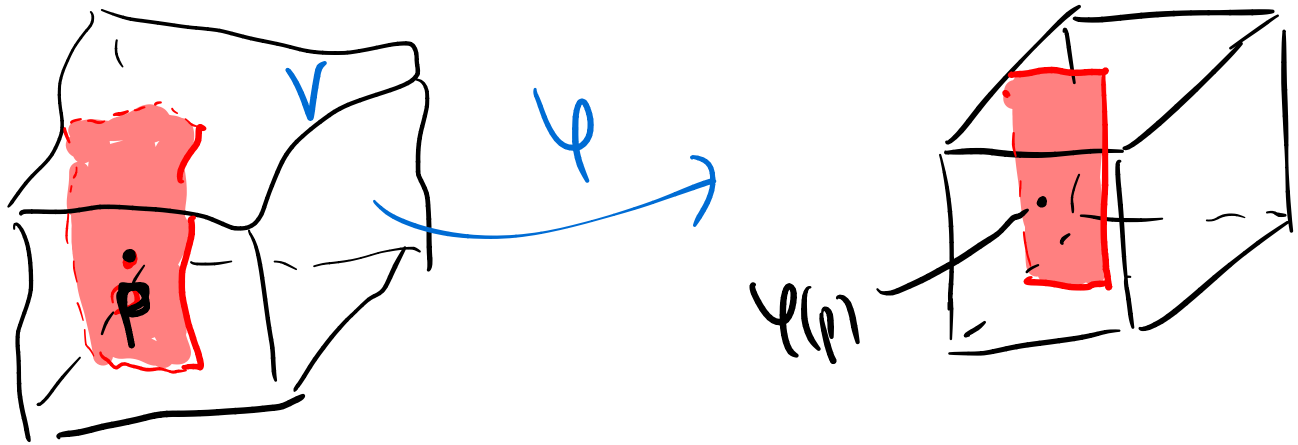
Proof (1) LHS $\overset{\text{Fubini}}{=} \displaystyle\int_{[-L,L]^{m-1}} dx^2 \cdots dx^m \int_{-L}^{0} dx^1 \frac{\partial f}{\partial x^1} \overset{\text{Fund thm calculus}}{=} \int_{[-L,L]^{m-1}} dx^2 \cdots dx^m f(0, x^2, \dots, x^m)$

(2) Exercícios

Def'n A subset $M \subset \mathbb{R}^n$ is a m-dim orientable submanifold with boundary of $\mathbb{R}^n$ if $\forall p \in M$, $\exists V \subset \mathbb{R}^n$ open nbhd of p an a positive diffeomorphisme $\varphi : V \to U$ onto an open set $U \subset \mathbb{R}^n$ such that

$\det(d\varphi_p) > 0$ $\forall p, (V, \varphi)$

$$\varphi(M \cap V) = (\mathbb{R}^m \times \{0\}) \cap U \cap \{x^1 \leq 0\}$$



$$\partial M := \{ p \in M : \varphi(p) \in \{x^1 = 0\} \cap \{\mathbb{R}^m \times \{0\}\})$$

smoth manifol of dim $m-1$

Def'n Let $\omega \in \Omega^m(\mathbb{R}^n)$ and $M \subset \mathbb{R}^n$ m-dim orientable submfld (possibly with ∂). We say that ω is integrable over M if $\exists$ $(V_\alpha, \varphi_\alpha)$ submanifold "atlas"

($\text{i.e } \bigcup_\alpha V_\alpha \supset M$)

and λ_α partition of unity subordn. to $\{V_\alpha\}$ st

$$\sum_\alpha \int_{W_\alpha \times \{0\}} \left| (\varphi_\alpha^{-1})^* (\lambda_\alpha w)(e_1, \dots, e_m) \right| dx^1 \dots dx^m < \infty$$

$W_\alpha \times \{0\} := (\mathbb{R}^m \times \{0\}) \cap U_\alpha \cap \{x^1 \leq 0\}$ $\qquad W_\alpha \subset \mathbb{R}^m$

If w is integrable over M, then:

$$\int_M w := \sum_\alpha \int_{W_\alpha \times \{0\}} (\varphi_\alpha^{-1})^* (\lambda_\alpha w)(e_1, \dots e_m)\, dx^1 \dots dx^m$$

first m vec. of canonic basis of $\mathbb{R}^n$

→ Let us check if only depends on M, ω

1. ω has cpt. support in V and

$$\begin{tikzcd} & U \quad (x^1, \dots, x^n) \\ V \arrow[ur, "\varphi"] \arrow[dr, "\tilde{\varphi}"'] & \\ & \tilde{V} \quad (y^1, \dots, y^n) \end{tikzcd}$$

$\bar{x} := (x^1, \dots, x^n)$

$\bar{y} := (y^1, \dots, y^n)$

$$\int_{\{\bar{x} \,:\, (\bar{x},0) \in U,\ x^1 \leq 0\} =: W \subset \mathbb{R}^m} (\varphi^{-1})^* \omega_{(\bar{x},0)} (e_1, \dots, e_m)\, d\bar{x} \overset{?}{=}$$

$$\overset{?}{=} \int_{\{\bar{y} \,:\, (\bar{y},0) \in \tilde{U},\ y^1 \leq 0\} =: \tilde{W} \subset \mathbb{R}^n} (\tilde{\varphi}^{-1})^* \omega_{(\bar{y},0)} (e_1, \dots, e_m)\, d\bar{y}$$

Let $\Psi = \tilde{\varphi} \circ \varphi^{-1} : U \longrightarrow \tilde{U}$

Notice $\Psi(\bar{x}, 0) \in \mathbb{R}^m \times \{0\}$

So, we can define $\hat{\Psi} : W \longrightarrow \tilde{W}$ as $\Psi(\bar{x}, 0) = (\hat{\Psi}(\bar{x}), 0)$

By def'n of pull-back

$\varphi^{-1} = \tilde{\varphi}^{-1} \circ \Psi$

$$(\varphi^{-1})^* \omega_{\underbrace{(\bar{x},0)}_{x}} (e_1, \dots, e_m) = \left((\tilde{\varphi}^{-1})^* \omega\right)_{\underbrace{(\hat{\Psi}(\bar{x}),0)}_{\Psi(x)}} \overbrace{\left(d\Psi_{(\bar{x},0)}(e_1), \dots, d\Psi_{(\bar{x},0)}(e_m)\right)}^{\left(d\hat{\Psi}_{\bar{x}}(e_1), \dots, d\hat{\Psi}_{\bar{x}}(e_m)\right)}$$

m-forms are multilinear and alternating hence, determinants

$$= \det(d\hat{\Psi}_{\bar{x}}) \left((\tilde{\varphi}^{-1})^* \omega\right)_{(\hat{\Psi}(\bar{x}),0)} (e_1, \dots, e_m)$$

Therefore,

$$\int_W ((\varphi^{-1})^*\omega)_{(\hat{x},0)}(e_1,\dots e_m)\, d\bar{x} =$$

$$\overset{(*)}{=} \int_W \underbrace{\det(d\hat{\varphi}_{\bar{x}})\, d\bar{x}}_{d\bar{y}} \; \underbrace{((\tilde{\varphi}^{-1})^*\omega)_{(\hat{\varphi}(\bar{x}),0)}(e_1,\dots e_m)}_{f(\underbrace{\hat{\varphi}(\bar{x})}_{\bar{y}})}$$

$\psi(\tilde{W}) =$

where

$$f(\bar{y}) := ((\tilde{\varphi}^{-1})^*\omega)_{(\bar{y},0)}(e_1,\dots,e_m)$$

$$\bar{y} = \hat{\varphi}(\bar{x})$$

standard change of variables formula

$$\overset{\text{RF}}{=} \int_{\widetilde{W}} \left(\left((\varphi^{-1})^* \omega \right)_{(\bar{y},0)} (e_1, \dots, e_m) \right) d\bar{y}$$

$\blacksquare$

2. (exercise) $\int_M \omega$ independent of $(V_\alpha, \varphi_\alpha)$, λ_α

$\{\lambda_\alpha\}_\alpha$, $\{\mu_\beta\}_\beta$ part. unity $\Rightarrow$ $\{\lambda_\alpha \mu_\beta\}_{\alpha,\beta}$

"Natural" orientation of ∂M (M oriented)

$p \in \partial M$, $p \in V$ $\varphi : V \longrightarrow U$

$\varphi(p) = (0, x^2, \dots x^m, 0, \dots 0)$ (def'n of ∂M)

$$\{ d(\varphi^{-1})_{\varphi(p)}(e_i) : 2 \leq i \leq m \}$$

define as positive basis of $T(\partial M)_p$

<u>Thm</u> (generalized Stokes) $M \subset \mathbb{R}^n$ orientable m-dim submfld with ∂, $\omega \in \Omega^{m-1}(M)$ st

$$\begin{cases} \omega \text{ integrable on } \partial M \\ d\omega \text{ integrable on } M \end{cases}$$

Then,

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof $\boxed{\text{Case 1}}$ ω cpt spt in V $\quad \varphi : V \to U$

is submfld chart

$$\int_{M \cap V} d\omega \overset{\text{def}}{=} \int_{\mathbb{R}^m \times \{0\} \cap \{x^1 \leq 0\}} (\varphi^{-1})^* d\omega \, (e_1, \dots, e_m) \, dx^1 \dots dx^m$$

pullback commutes with d

$$= \int_{\mathbb{R}^m \times \{0\} \cap \{x^1 \leq 0\}} d \underbrace{(\varphi^{-1})^* \omega}_{\bar{\omega}} \, (e_1, \dots, e_m) \, dx^1 \dots dx^m$$

$\bar{\omega}$ is $(m-1)$ form in $U \subset \mathbb{R}^n$

$$\bar{\omega} = \sum_I \bar{\omega}_I \, dx^I \qquad (I = (i_1, \dots, i_{m-1}))$$

$$d\bar{\omega} = \sum_I \sum_i \frac{\partial}{\partial x^i} \bar{\omega}_I \, dx^i \wedge dx^I$$

Notice $J = (j_1, \dots, j_m) \quad j_1 < j_2 < \dots < j_m$

$$dx^J(e_1, \dots, e_m) = \begin{cases} 1 & (j_1, \dots, j_m) = (1, 2, \dots, m) \\ 0 & \text{otherwise} \end{cases}$$

$$d\overline{w} \overset{(i)}{=} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} f^i \, dx^1 \wedge \dots \wedge dx^m + \text{other term which give } 0 \text{ when evaluated at } (e_1, \dots, e_m)$$

$$\overline{w} \overset{(ii)}{=} \sum_{i=1}^{m} (-1)^{i-1} f^i \, dx^1 \wedge \dots \wedge \overset{\text{OUT}}{\widehat{dx^i}} \wedge \dots \wedge dx^m$$

$$+ \text{ other terms as above}$$

Therefore,
$$\int_M d\omega \overset{(i)}{=} \sum_{i=1}^{m} \int_{\{x^1=0\}} \frac{\partial}{\partial x^i} f^i(x^1,\dots,x^n,0,\dots 0)\, dx^1 \dots dx^m$$

$$\overset{\text{baby Stockes}}{=} \int_{\{x^1=0\}} f^1(\underset{\substack{\| \\ 0}}{x^1},\dots x^n,0\dots 0)\, dx^2 \dots dx^n$$

$$\overset{(ii)}{=} \int_{\{x^1=0\}} \bar{\omega}(e_2,\dots,e_m)$$

$$= \int_{\partial M} \omega$$

$\boxed{\text{case } 2}$ $\quad \omega = \sum_\alpha \lambda_\alpha \omega^\alpha$

λ_α partition of id subordinated to $\{V_\alpha\}$

$(V_\alpha, \varphi_\alpha)$ submfld Atlas

Then
$$\int_M d\omega = \sum_\alpha \int_M d(\lambda_\alpha \omega^\alpha)$$
$$\overset{\boxed{\text{Case 1}}}{=} \sum_\alpha \int_{\partial M} \lambda_\alpha \omega^\alpha = \int_{\partial M} \omega$$

$\blacksquare$

Ch 9 (Sard's thm & mapping degree)

A set $A \subset \mathbb{R}^m$ has **measure zero** or is a **null set** if $\forall \varepsilon > 0$ $\exists$ seq. of **cubes** $Q_i \subset \mathbb{R}^m$ s.t. $A \subset \bigcup_i Q_i$

$\sum_i |Q_i| < \varepsilon$ $\quad$ ($Q = [x^1, x^1+r] \times [x^2, x^2+r] \times \cdots \times [x^m, x^m+r]$)

$$|Q| = r^m$$

The union of countably many null sets is a null set

[usual measure thy trics] $\quad \varepsilon/2, \quad \varepsilon/2^2, \quad , \quad , \quad \varepsilon/2^k \xrightarrow[\text{sum}]{} \varepsilon$

If $V \subset \mathbb{R}^n$ is open $F : V \to \mathbb{R}^n$ C^1

$A \subset V$ null $\Rightarrow$ $F(A)$ null

Pf. $V = \bigcup_{k=1}^{\infty} B_k$ $\overline{B_k}$ cpt ball, consider $A \cap B_k$

an covers $A \cap \overline{B_k} \subset \bigcup_i Q_i \subset K \subset V$
cpt set fixed

$F \in C^1 \Rightarrow F|_K$ lipschitz

$\Rightarrow$ $\operatorname{diam}(F(Q_i)) \leq L \operatorname{diam}(Q_i)$

(sum over i)

$\Rightarrow$ $F(A \cap B_k)$ is null

9.1 **Def** A subset A of a C^r mfld M, $r \geq 1$, has measure zero if $\forall$ chart (φ, U) of M

null $\;\; \varphi(A \cap U)$ has measure 0

$A \subset M$ null $\iff \varphi(A \cap U)$ null for all charts in some atlas

9.2 **Thm** (Morse 1939, Sard 1942) "Sard's lemma"

If $F: M^m \longrightarrow N^n$ is a C^r map with $r > \max\{0, m-n\}$ then the set of singular values of F has meas. zero.

pf

$$\begin{tikzcd} M^m \arrow[r, phantom, ""] & & N^n \\ U \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, "F"] \arrow[d, "\varphi"] & & V \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, "\psi"] \\ \mathbb{R}^m \supset \varphi(U) \arrow[rr, "f"] & & \psi(V) \subset \mathbb{R}^n \end{tikzcd}$$

$m \geq n$

$F, f \in C^\infty$

We want to show given a cube

$$Q_r(x_0) = x_0 + (-r, r)^m \subset \mathbb{R}^m$$

$$\forall\, Q_r(x_0) \subset \varphi(U)$$

Let us show $\Sigma := \{ x \in \overline{Q_{r/2}(x_0)} \mid df_x = 0 \}$

$(r > 0)$

satisfies $f(\Sigma)$ null

$$\Sigma^* := \left\{ x \in \Sigma \;\middle|\; \frac{\partial^\alpha f}{\partial x^\alpha}(x) = 0 \quad \forall \alpha = (\alpha_1, \dots \alpha_m),\ |\alpha| \leq k \right\}$$

since $f|_{\overline{Q_{\frac{3r}{4}}(x_0)}}$ is uniformly C^k

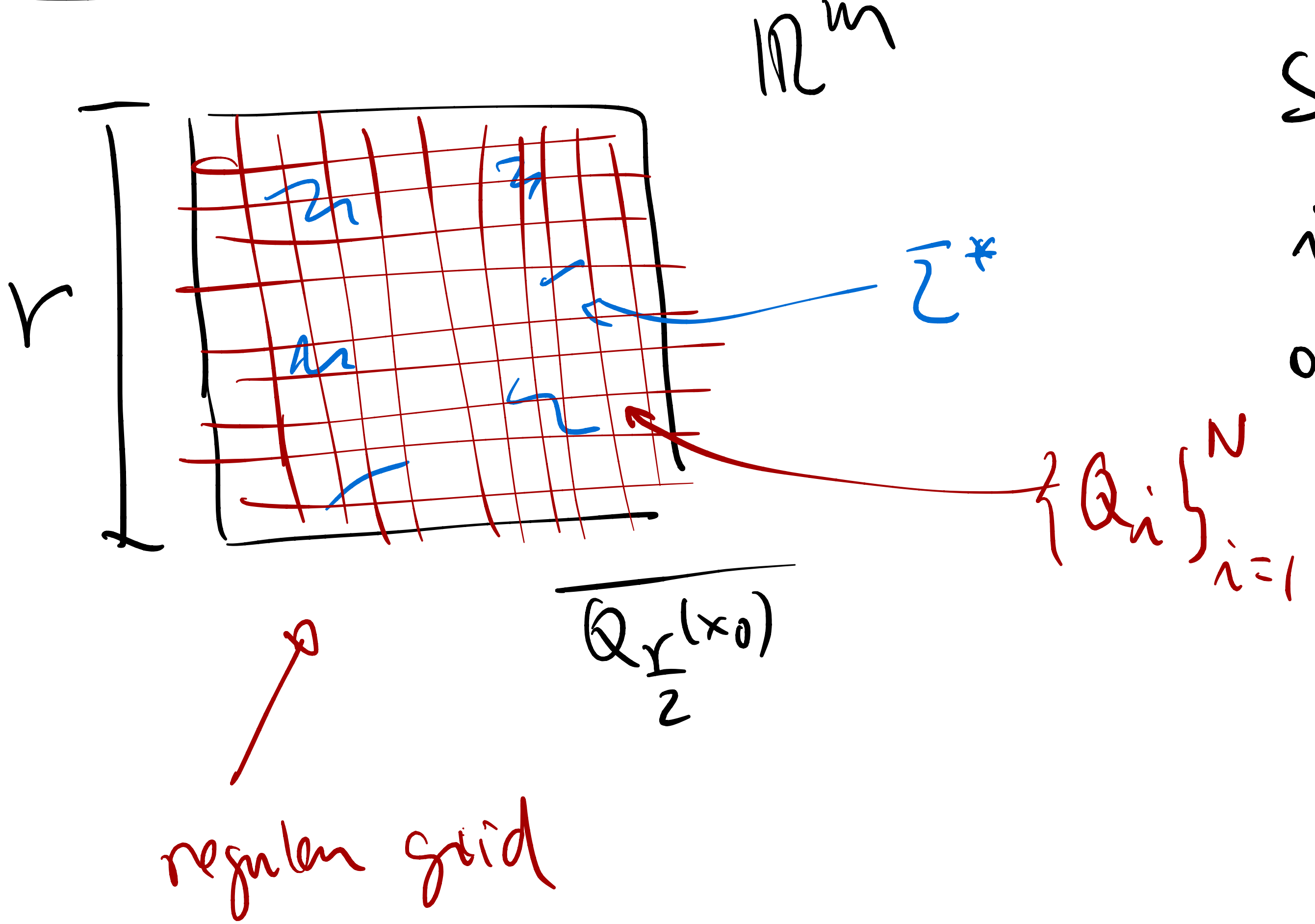
$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon)$, $\forall x, x' \in Q_{r/2}(x_0)$

$$(*) \quad \Big| f(x') - \underbrace{f(x) - \sum_{1 \leq |\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\,(x'-x)^\alpha}_{\text{Taylor expansion}} \Big| \leq \varepsilon\, |x'-x|^k$$

$|x'-x| < \delta$

$(*) \Rightarrow \forall x \in \Sigma^*$

$$(**) \quad |f(x') - f(x)| \leq \varepsilon |x'-x|^k, \quad |x'-x| < \delta$$



Split $\overline{Q_{r/2}(x_0)}$ into smaller cubes of diameter $\rho > 0$

$$N \leq C(m) \left(\frac{r}{\rho}\right)^m$$

<u>if</u> $\Sigma^* \cap Q_i \neq \emptyset$ $\quad \exists\, x_i \in \Sigma^* \cap Q_i$

$(**) \Rightarrow$

$$f(\Sigma^* \cap Q_i) \subset B_{f(x_i)}(\varepsilon \rho^k) \subset \text{cube of side } 2\varepsilon\rho^k =: Q_i' \subset \mathbb{R}^n$$

$$|f(\Sigma^* \cap Q_i)| \leq \sum_{i=1}^{N} |Q_i'| = N(\varepsilon\rho^k)^n$$

$$\lesssim C(m)\left(\frac{r}{\rho}\right)^m (\rho^k)^n \lesssim C(m,r)\,\rho$$

$\rho^{-m} \rho^{kn} \le \rho^{kn-m} \le \rho$ $\quad$ k large $\quad$ $\rho \in (0,1)$

$\Updownarrow$ $kn - m \geq 1$

$\Rightarrow f(\Sigma^*)$ is null set

By defin of Σ^*, if $x \in \Sigma \setminus \Sigma^*$

- $df(x)$ does not have maximal rank
- $\exists \alpha$ $|\alpha| = \alpha_1 + \dots + \alpha_m \le K$ st

$$\frac{\partial f}{\partial x^\alpha}(x) \neq 0$$ with $|\alpha|$ minimal

$$\Rightarrow \quad x \in \Gamma_\alpha := \left\{ x \in \varphi(U) := \frac{\partial f}{\partial x^\alpha} > 0 \text{ and } \frac{\partial f}{\partial x^\beta} = 0 \quad \forall \ |\beta| < |\alpha| \right\}$$

Example

$$\boxed{\Gamma_\alpha = \left\{ x : \frac{\partial f}{\partial x^1 \partial x^2} > 0 \, , \; \frac{\partial f}{\partial x^1} = \frac{\partial f}{\partial x^2} = 0 \right\} \qquad \varphi(U) \subset \mathbb{R}^2}$$

$\Rightarrow \quad \Sigma \setminus \Sigma^* \subset$ Union of codimension ≥ 1 submanifolds of $\varphi(U)$

induction over m

$$M^{m} \longrightarrow N^n$$

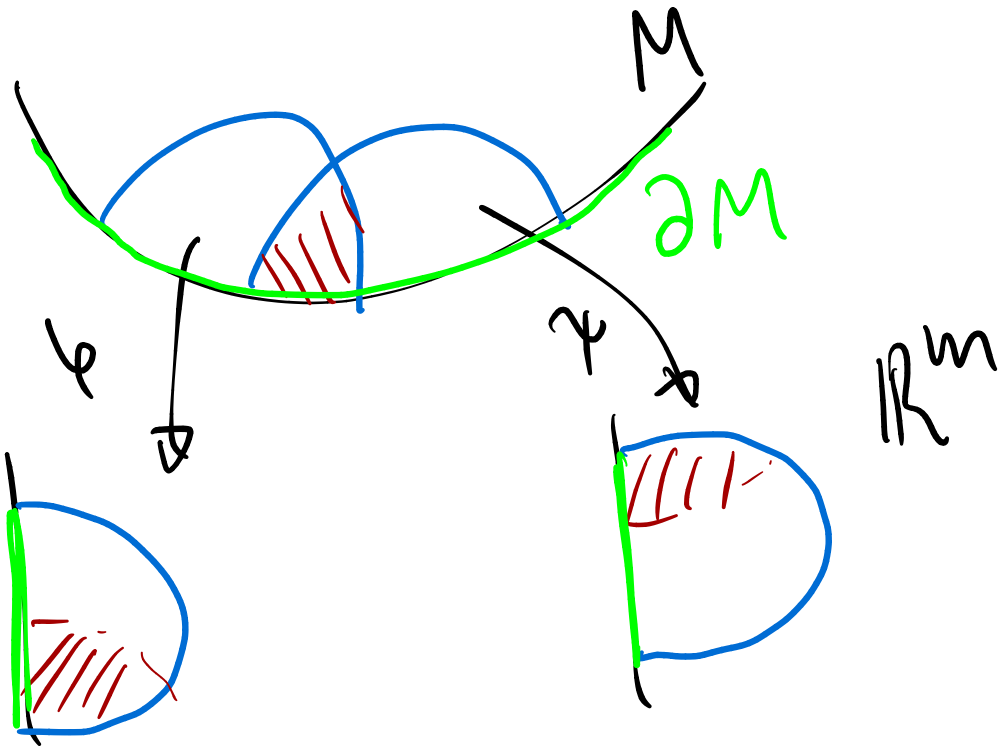
Apply result for C^∞ maps $M^{m-1} \longrightarrow N^n$

$\blacksquare$

Manifold with bdry

m-dim C^∞ mflds with bdry are defined as in 8.1, except that the images of charts $\varphi(U) \subset \mathbb{R}^m$ are open in a halfspace $= \{x \in \mathbb{R}^m \mid x^1 \geq 0\} = H$

$\{x \in \mathbb{R}^m \mid x^1 = 0\} = \partial H$



Bdry of M

$\partial M := \{ p \in M \mid \varphi(p) \in \partial H,$ for some (hence every) chart $\varphi \}$

Tangent space

Let M^m be C^∞ mfd with ∂, for $p \in M$, TM_p is defined as in 8.4

Notice that if $p \in \partial M$ $d(\psi \circ \varphi^{-1})_{\varphi(p)}$ still $\exists$, so we can still define equivalence classes at these pts.

For $p \in \partial M$, $T(\partial M)_p$ is in a canonical way a subspace of TM_p

$$\left(\{ [\varphi, \xi]_p \mid \xi \in T(\partial H)_{\varphi(p)} \text{ for } \varphi : U \to \varphi(U) \subset H \right)$$

The differential $dF_p : TM_p \longrightarrow TN_{F(p)}$ is defined exactly as in ch. 8.

<u>9.3 Thm</u> (regular value thm for mflds with ∂)
N^n mfd with ∂, Q^k mfd, $F : N \longrightarrow Q$ C^∞
If $q \in F(N)$ is a <u>regular value</u> of $F|_{N - \partial N}$ as well as of $F|_{\partial N}$, then $M := F^{-1}\{q\}$ is a C^∞ mfld with ∂, $\dim M = n-k$, and

$$\partial M = M \cap \partial N$$

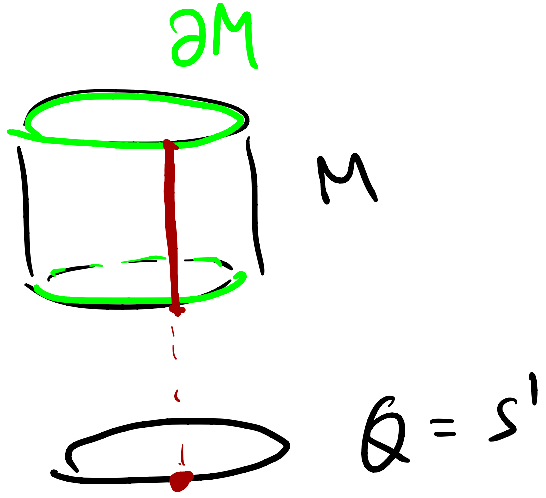
<u>Pf.</u> Applications of 8.7

Examples

$N = S^1 \times [0,1] \subset \mathbb{R}^3$

$Q = S^1$

$F(x) = (x^1, x^2)$



9.4 Thm

If M is a cpt. C^∞ mfld with ∂, there exists no smooth retraction $F : M \to \partial M$ (i.e. $F(p) = p \ \forall p \in \partial M$)

Pf. (Hirsch) Indirect. $F : M \to \partial M$ C^∞ retraction

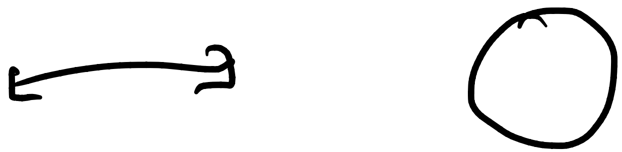
By Sard's thm $\exists$ a regular value q of $F|_{M \setminus \partial M}$,

q is also reg. val of $F|_{\partial M}$

9.3 $\Rightarrow$ $F^{-1}\{q\}$ is <u>compact</u> 1-dim manifold (with ∂)

$\partial(F^{-1}\{q\}) = F^{-1}\{q\} \cap \partial M = \{q\}$ (F is retraction)

Every cpt 1-dim mfld with ∂ is a finite union



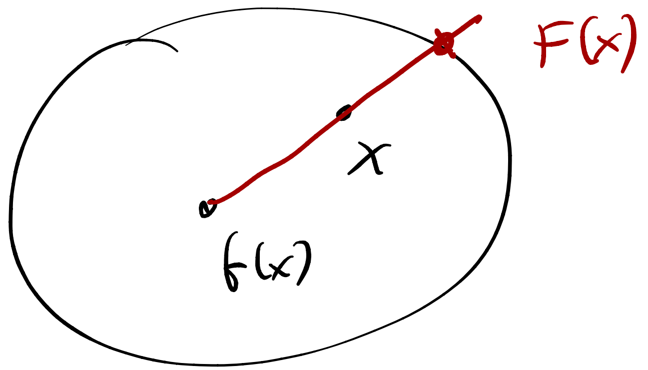
$\Rightarrow$ the number of bdry points of $\partial(F^{-1}\{q\})$ must be <u>even</u> !

$\blacksquare$

Corollary (Brouwer's fixed pt thm)

Every cont map $G : B^m \to B^m = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ has a fixed pt

Pf. If we had (by contr.) $G : B^m \to B^m$ without fixed pt.

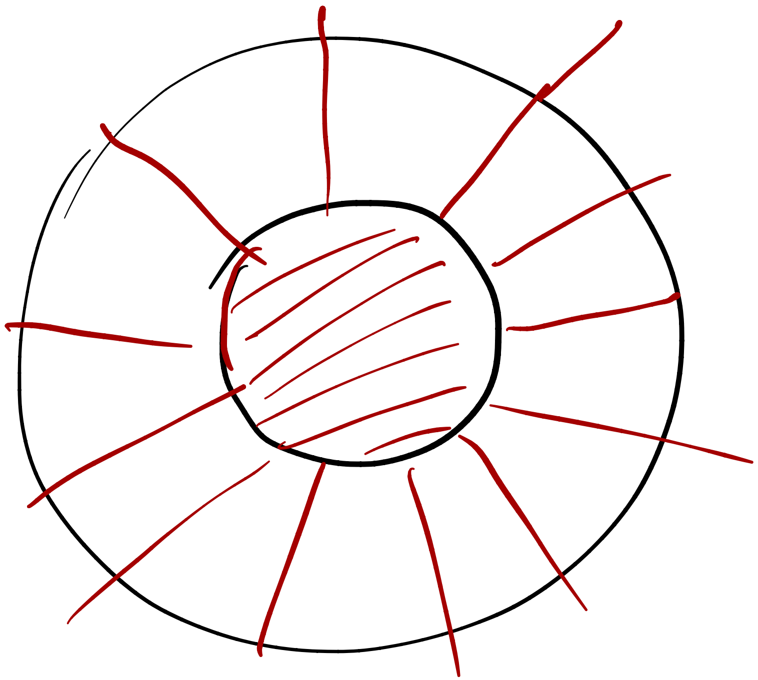


F would be C^0 retraction $B^m \to S^{m-1}$

We can smooth $F \rightsquigarrow \tilde{F}$, C^∞ retraction

$$F'(x) = \begin{cases} F(2x) & x \in \overline{B_{1/2}} \\ F\left(\frac{x}{|x|}\right) & x \in B_{3/2} \setminus B_{1/2} \end{cases}$$

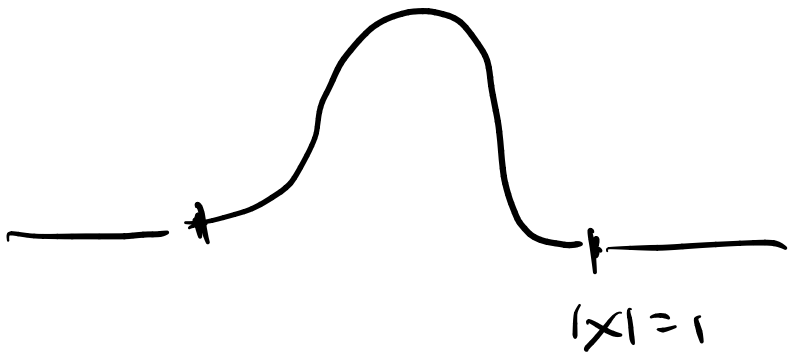
← ball of radius 1/2 centered at 0



$$\tilde{F} = F' * \eta_\varepsilon(|x|)$$

$$\eta(|x|) = e^{-\frac{1}{1-|x|^2}} \frac{1}{Ct}$$

$$\eta_\varepsilon = \frac{1}{\varepsilon^m} \eta\left(\frac{x}{\varepsilon}\right)$$



Mapping degree

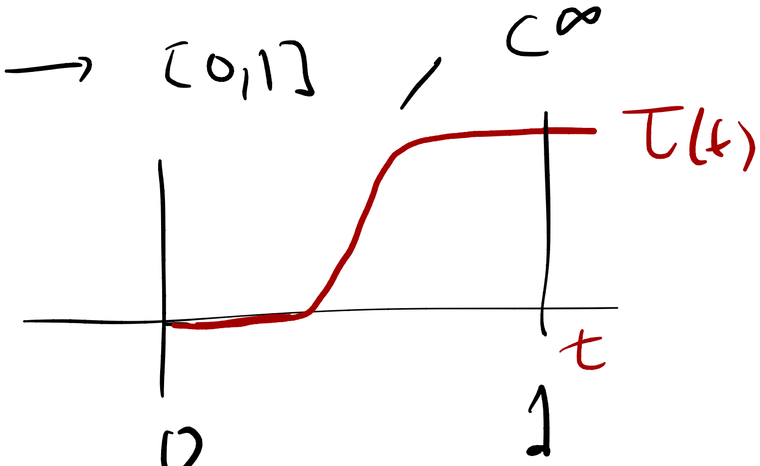
M, N mflds $\quad F, G : M \longrightarrow N \quad C^\infty$

A C^∞ map $H : M \times [0,1] \to N$ with $H(\cdot, 0) = F$ and $H(\cdot, 1) = G$ is called smooth homotopy from F to G.

If in addition $H(\cdot, t) : M \longrightarrow N$ is C^∞ diffeo $\forall t \in [0,1]$ then H is called smooth isotopy

$F \sim G$ smoothly homotopic (if $\exists$ smooth homotopy) is an equivalence relation

For transitivity use $\tau : [0,1] \to [0,1]$



$$\tau(t) = \begin{cases} 0 & t \in [0, \frac{1}{3}] \\ 1 & t \in [\frac{2}{3}, 1] \end{cases}$$

"compose" two given homotopies using τ

<u>9.6 Lemma</u> N connected mfd, $q, q' \in N$

$\Rightarrow$ $\exists$ smooth isotopy $H : N \times [0,1] \to N$

s.t. $H(\cdot, 0) = id_N$ and $H(q,1) = q'$

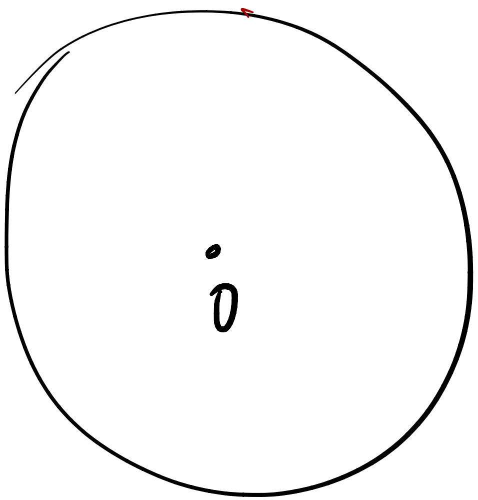
<u>Pf</u> Show first $\forall\, y \in B_1^n := \{x \in \mathbb{R}^n \mid |x| < 1\}$

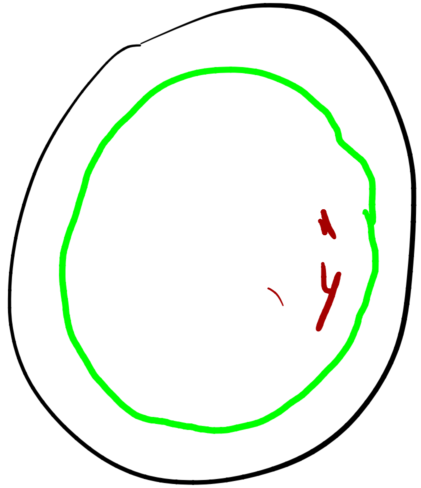
$\exists$ smooth isotopy $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$

s.t. $H(z,t) = z \quad \forall\, z \in \mathbb{R}^n \setminus B_1^n \quad \forall\, t \in [0,1]$

$H(\cdot, 0) = \mathrm{id}_{\mathbb{R}^n}$

$H(0,1) = y$





Choose $\lambda \in C_c^\infty(\mathbb{R}^n)$ s.t.

$$\lambda(x) = \begin{cases} 1 & |x| \leq |y| < 1 \\ 0 & |x| \geq 1 \end{cases} \qquad (y \in B_1^n)$$

vector field $X(x) := y \cdot \lambda(x) \rightsquigarrow \{\varphi^t\}_{t \in \mathbb{R}}$ associated flow

$\varphi^1(0) = y \Rightarrow$ Put $H(z,t) := \varphi^t(z)$

$$\frac{d}{dt}\varphi = \varphi \circ X$$

For connected mfld N, define equivalence relation between its pts $q \sim q' \iff \exists$ isotopy as claimed

By result in B_1^n (taking charts), equivalence classes are open

$\Rightarrow N$ is split into disjoint open sets by equivalence classes $\Rightarrow$ only 1 equivalence class!

$\square$

Now: M, N mflds of same dimension M cpt., N connected

If $F : M \longrightarrow N$ is C^∞ and $q \in N$ is reg. value then $F^{-1}\{q\}$ is a finite set (cpt 0-dim submfld), possibly $\emptyset$

9.7. Thm M, N as above

(1) $F, G : M \to N$ smoothly homotopic, q reg. value of both F and G $\Rightarrow$ $\# F^{-1}\{q\} \equiv \# G^{-1}\{q\} \mod 2$

(2) $F : M \to N$ C^∞ q, q' two reg. values of F

$$\Rightarrow \quad \# F^{-1}\{q\} \equiv \# F^{-1}\{q'\} \mod 2$$

The number $\deg_2(F) := (\# F^{-1}\{q\} \mod 2) \in \{0,1\}$ of F is called <u>mapping degree modulo 2</u> of F

- by (2) indep. of q
- by (1) invariant under smooth homotopy

Pf $\quad M\times\{0\} \cup M\times\{1\} \underset{(*)}{=} \partial(M\times[0,1])$

if $H: M\times[0,1] \longrightarrow N$ is a smooth homotopy $\begin{cases} H(\cdot,0)=F \\ H(\cdot,1)=G \end{cases}$ $(*)$

$\Longrightarrow$ q is reg value of $H|_{\partial(M\times[0,1])}$

by assumption

In order to apply 9.3, suppose first q is reg. value of $H|_{M\times(0,1)}$

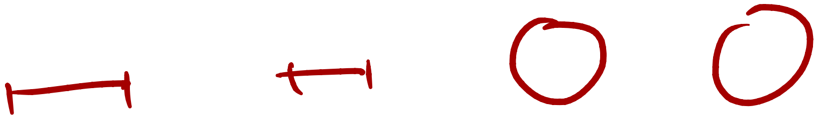
$\overset{9.3}{\Longrightarrow}$ $H^{-1}\{q\}$ is cpt. 1-dim mfd (possibly $\emptyset$)

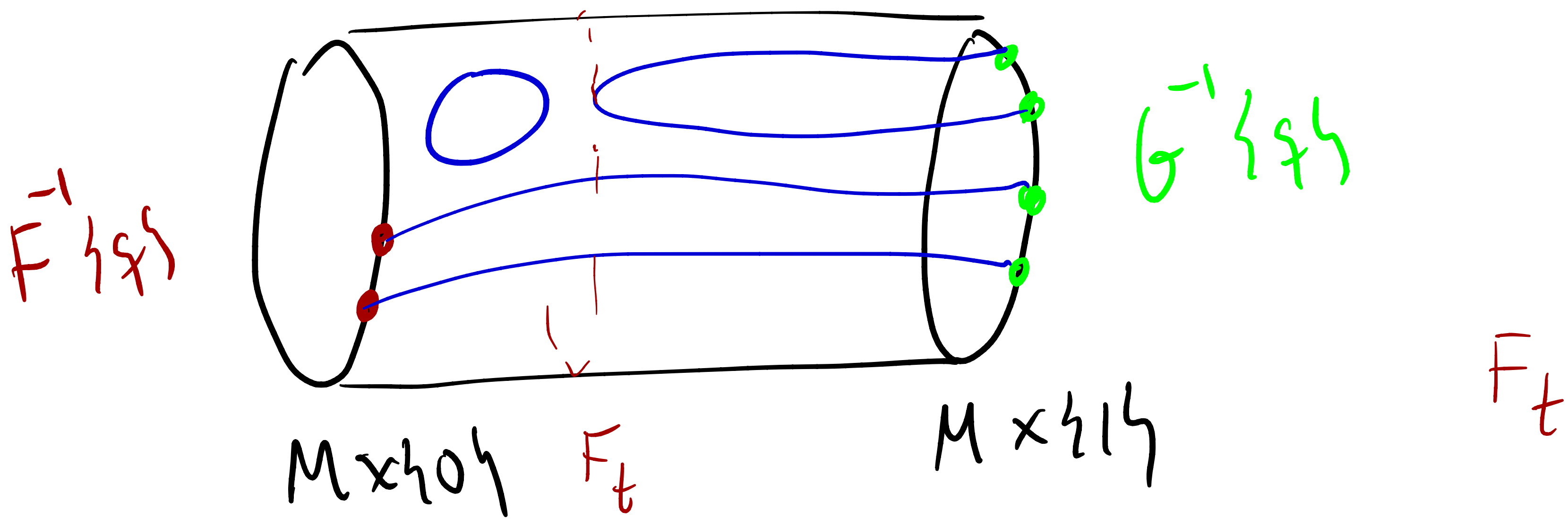
with bdry

$$\partial H^{-1}\{q\} = H^{-1}\{q\} \cap \partial(M\times[0,1]) \overset{(*)}{=} F^{-1}\{q\}\times\{0\} \cup G^{-1}\{q\}\times\{1\}$$

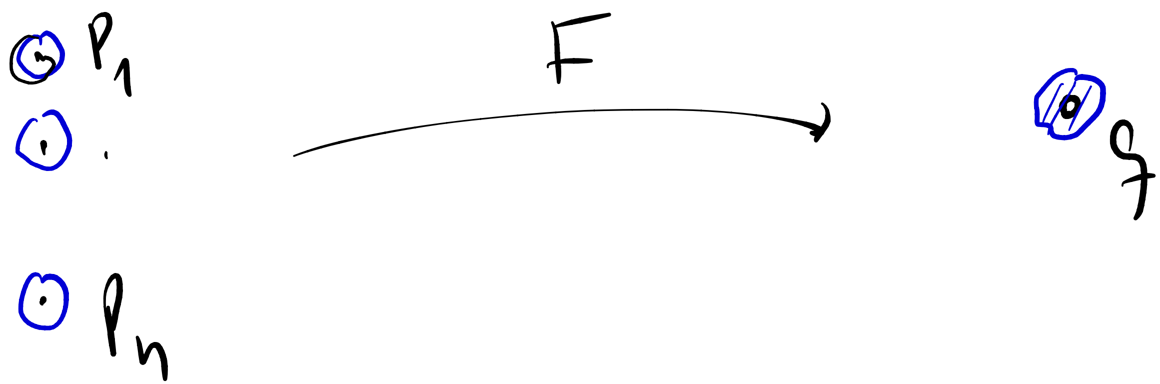
Hence, taking #

$$\overbrace{\#(\partial \underbrace{H^{-1}\{q\}}_{\text{1-dimensional cpt mfld with } \partial})}^{\text{even}} = \#(F^{-1}\{q\}) + \#(G^{-1}\{q\})$$





General case: notice $\exists$ open nbhd V of q in N s.t. all $q' \in V$ are regular values of both F and G, and, in addition, $\# F^{-1}\{\cdot\}$ and $\# G^{-1}\{\cdot\}$ are constant in V (exercise)



But then, by Sard's Lemma $\exists\, q' \in V$ st is reg. value of $H|_{M \times (0,1)}$

$$\# F^{-1}\{q\} = \# F^{-1}\{q'\} \underset{\text{mod} 2}{\equiv} \# G^{-1}\{q'\} = \# G^{-1}\{q\}$$

this finishes the pf of (1)

C^∞ homotopic

(2) Let $G : N \to N$ be diffeomorphism $\sim id_N$

s.t. $G(q) = q'$ (Lemma 9.6)

$\Longrightarrow$ q' reg. value of $G \circ F$

$$\left[\begin{array}{l} d(G\circ F)_p = dG_q \circ dF_p \\ \text{surjective } \forall p \in (G\circ F)^{-1}\{q'\} \\ \qquad = F^{-1}\{q\} \end{array}\right.$$

$$G \circ F \sim F \overset{(1)}{\Longrightarrow} \# \underbrace{(G\circ F)^{-1}\{q'\}}_{\substack{\shortparallel \\ F^{-1}\{q\}}} \equiv F^{-1}\{q'\} \text{ mod } 2$$

$\blacksquare$

<u>Example</u> 9.7 (1) $\Rightarrow$ $\nexists$ no C^∞ retraction $F: B^m \to \underbrace{\partial B^m}_{S^{m-1}}$

since otherwise $H: S^{m-1} \times [0,1] \to S^{m-1}$

$$H(p,t) = F(tp)$$

$$H(\cdot,0) \equiv F(0) \quad (\deg_2 \equiv 0)$$

$$H(\cdot,1) \equiv \mathrm{id}_{S^{m-1}} \quad (\deg_2 \equiv 1)$$

If M, N <u>oriented</u>, $\dim M = \dim N$, M cpt, N connected

then the <u>mapping degree</u> $\deg(F) \in \mathbb{Z}$ of a C^∞

map $F: M \longrightarrow N$ is defined as

$$\deg(F) = \sum_{p \in \underbrace{F^{-1}\{q\}}_{\text{finite}}} \operatorname{sgn}(dF_p) \qquad (q \text{ regular})$$

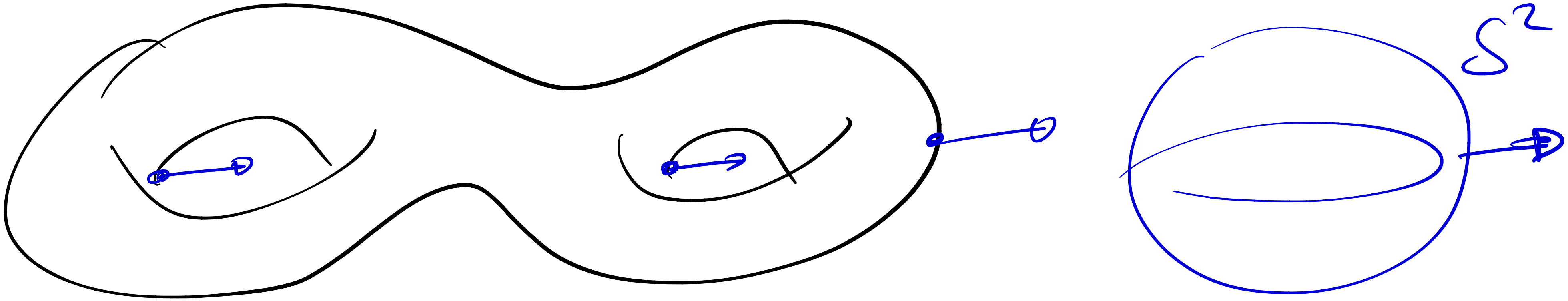
similarly as in mod 2 case one shows that

$\deg(F)$ does not depend on q

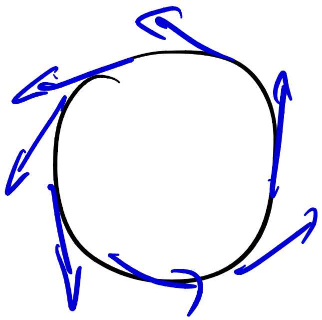
Applications (exercise) $M \subset \mathbb{R}^3$ cpt connected surface

$F : M \longrightarrow S^2$ is exterior Gauss map

$$\deg(F) = \frac{1}{2} \chi(M)$$



9.8 Thm S^m admits a nowhere vanishing C^∞ tangent vector field if, and only if, m is odd

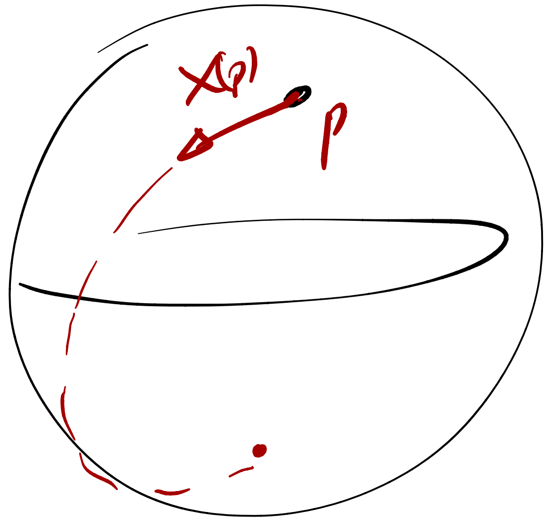


Pf suppose $X : S^m \to S^m$ C^∞ tangent v.f also $\frac{X}{|X|}$ is (assume wlos $|X| = 1$)

$$H : S^m \times [0,1] \longrightarrow S^m \qquad |p| = 1$$

$$H(p,t) = p\cos(\pi t) + X(p)\sin(\pi t)$$

is C^∞ homotopy from id to $-$id



But $\begin{cases} \deg(id) = 1 \\ \deg(-id) = (-1)^{m+1} \end{cases}$

So by homotopy inv. of deg

$\Rightarrow \quad 1 = (-1)^{m+1} \quad \Rightarrow \quad m$ odd

Conversely if $m = 2K-1$ odd $\quad S^{2K-1} \subset \mathbb{R}^{2K}$

$$X(p^1, \dots, p^{2K}) = (p^2, -p^1, p^4, -p^3, \dots, p^{2K}, -p^{2K-1})$$

defines a nonzero tangent v.f on S^m $\blacksquare$

$\xi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad C^\infty$ v.f. $\quad \xi(x) \neq 0$ for $0 < |x| \leq 1$

$F : S^1 \to S^1 \qquad F(x) = \dfrac{\xi(x)}{|\xi(x)|}$

$\deg F = I_\xi(0)$ Poincaré index of ξ at 0