Exam Problems HS21

1. The Catenoid

Consider the following parametrization of a "cylindrical" surface of revolution in $\mathbb{R}^3$:

$$S_f \coloneqq \{x \in \mathbb{R}^3 \mid \sqrt{(x^1)^2 + (x^2)^2} = f(x^3),\ |x^3| \leq 1\},$$

where $f\colon [-1,1] \to (0,\infty)$ is smooth. Notice that ∂S_f is the union of two circumferences.

(a) Show that the area of S_f is given by $2\pi \int_{-1}^{1} f(t)\sqrt{1+f'(t)^2}\,dt$.

(b) Prove that if S_f has minimal area among all cylindrical surfaces as above with the same boundary, then f must satisfy:

(i) $ff'' = 1 + (f')^2$,

(ii) $\left(\frac{f}{\sqrt{1+(f')^2}}\right)' = 0$.

(c) Show that solutions of (ii) must be of the form $f(t) = a\cosh(\frac{t-b}{a})$ for some $b \in \mathbb{R}$ and $a > 0$.
[*Hint:* Use $\int \frac{dy}{\sqrt{y^2-a^2}} = \cosh^{-1}(\frac{y}{a}) +$ constant.]

(d) Prove that $\int_{S_f} K\,dA = -2\pi \int_{f'(-1)}^{f'(1)} \frac{dz}{(1+z^2)^{3/2}}$.

Solution: a) consider the parametrization $(s,t) \mapsto (\cos s f(t), \sin s f(t), t)$. The first fundamental form is

$$g_{11} = (-\sin s f(t), \cos s f(t), 0)\cdot(-\sin s f(t), \cos s f(t), 0) = f(t)^2$$
$$g_{12} = g_{21} = (-\sin s f(t), \cos x f(t), 0)\cdot(\cos s f'(t), \sin x f'(t), 1) = 0$$
$$g_{22} = (\cos s f'(t), \sin s f'(t), 1)\cdot(\cos s f'(y), \sin s f'(t), 1) = (1 + f'(t)^2)$$

Hence

$$\text{Area}(S_f) = \int_0^{2\pi} ds \int_{-1}^{1} dt \sqrt{\det(g_{ij})} = 2\pi \int_{-1}^{1} f(t)\sqrt{1+f'(t)^2}dt$$

b) Consider $\eta \in C_c^\infty(-1,1)$. If S_f has minimal area among such cylindrical surfaces then $0 = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{Area}(S_{f+\varepsilon\eta})$. That is,

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{-1}^{1} (f+\varepsilon\eta)\sqrt{1+(f'+\varepsilon\eta')^2}dt = \int_{-1}^{1}\left(\eta\sqrt{1+(f')^2} + f\frac{f'\eta'}{\sqrt{1+(f')^2}}\right)dt$$
$$= \int_{-1}^{1}\left(\eta\left(\sqrt{1+(f')^2} - \left(\frac{ff'}{\sqrt{1+(f')^2}}\right)'\right)\right)dt,$$

where we integrated by parts and used that $\eta(-1) = \eta(1) = 0$.

Since η is arbitrary we find

$$\begin{aligned}
0 &= \left(\sqrt{1+(f')^2} - \left(\frac{ff'}{\sqrt{1+(f')^2}}\right)'\right. \\
&= \frac{1+(f')^2}{\sqrt{1+(f')^2}} - \frac{(f')^2}{\sqrt{1+(f')^2}} - \frac{ff''}{\sqrt{1+(f')^2}} + \frac{f(f')^2f''}{(1+(f')^2)^{3/2}} \\
&= \frac{1}{\sqrt{1+(f')^2}} - \frac{(1+(f')^2)ff''}{(1+(f')^2)^{3/2}} + \frac{f(f')^2f''}{(1+(f')^2)^{3/2}} \\
&= \frac{1}{\sqrt{1+(f')^2}} - \frac{ff''}{(1+(f')^2)^{3/2}}
\end{aligned}$$

Thus we obtain $ff'' = 1 + (f')^2$. This implies

$$\left(\frac{f}{\sqrt{1+(f')^2}}\right)' = \frac{f'}{\sqrt{1+(f')^2}} - \frac{ff'f''}{(1+(f')^2)^{3/2}} = 0.$$

c) Integrating the previous ODE we obtain

$$\frac{f}{\sqrt{1+(f')^2}} = a,$$

where $a > 0$ (since $f > 0$). Hence, $f^2 = (1+(f')^2)a^2$ and $\frac{af'}{\sqrt{f^2-a^2}} = 0$

Integrating again (using the hint) we obtain $t = a\cosh^{-1}(\frac{f}{a}) + b$, where $b \in \mathbb{R}$. That is, $f(t) = a\cosh\left(\frac{t-b}{a}\right)$, as claimed.

d) Since S_f is a minimal surface its Gauss curvature K is nonpositive. Notice that the Gauss map is of the form

$$\nu(s,t) = \frac{(-\cos s, -\sin s, f')}{\sqrt{1+(f')^2}}.$$

Since the function $z \mapsto \frac{z}{\sqrt{1+z^2}}$ is increasing and $t \to f'(t)$ is also increasing (the hyperbolic cosine is convex) we obtain that the gauss map is injective (for $s \in [0, 2\pi)$, $t \in (-1, 1)$). Hence, since $-KdA$ is the Jacobian of the Gauss map, we have that $\int_{S_f} KdA$ equals the area of the image by ν of S_f (a subset of $\mathbb{S}^2$). If suffices to notice that ,by the previous considerations,

$$\nu(S_f) = \left\{ \frac{(\cos s, \sin s, z)}{\sqrt{1+z^2}} \ : \ s \in [0, 2\pi), \ z \in (f'(-1), f'(+1)) \right\}.$$

Using the parametrization of the sphere $(s,z) \mapsto \frac{(\cos s, \sin s, z)}{\sqrt{1+z^2}}$ we compute

$$g_{11} = \tfrac{1}{1+z^2}$$

$$g_{12} = g_{21} = 0$$

$$g_{22} = \tfrac{z^2}{(1+z^2)^3} + \left(\tfrac{1}{\sqrt{1+z^2}} - \tfrac{z^2}{(1+z^2)^{3/2}}\right)^2 = \left(\tfrac{1}{1+z^2}\right)^2$$

Therefore, the element of area of the sphere is $\frac{dsdz}{(1+z^2)^{3/2}}$ and hence

$$\int_{S_f} (-K)dA = \text{Area}(\nu(S_f)) = -\int_0^{2\pi} ds \int_{f'(-1)}^{f'(1)} \tfrac{dz}{(1+z^2)^{3/2}}.$$

2. Lie Bracket and Curvature

Let X be a C^∞ vector field on an open set $U \subset \mathbb{R}^n$. By the identification of vector fields and derivations, X acts on C^∞ functions:

$$Xf = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j} f \quad \text{for } f \in C^\infty(U),$$

where X^i are the components[1] of X. Similarly, if Y is a another C^∞ vector field on U, we let X act on Y component-wise. That is, we denote XY the vector field with components

$$(XY)^i := XY^i = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j} Y^i.$$

Given two smooth vector fields X, Y as above, define their *Lie bracket* $[X,Y] := Z$ as the map $Z\colon C^\infty(U) \to C^\infty(U)$ defined by

$$Zf := X(Yf) - Y(Xf).$$

(a) Show that Zf can be written as $\sum Z^i \frac{\partial}{\partial x^i} f$ and compute Z^i in terms of the X^i's and Y^i's. Deduce that Z is a smooth vector field on U.

(b) Prove $[X,Y] = XY - YX$.

[1] That is, $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ in the derivation point of view, or simply $X(p) = (X^1(p), X^2(p), \ldots, X^n(p)) \in \mathbb{R}^n \cong TU_p$ for all $p \in \mathbb{R}^n$.

(c) Let $\psi\colon U \to V$ be a C^∞ diffeomorphism. Define the *push-forward* of a vector field T on U, denoted $\psi_* T$, as

$$\psi_*(T)(q) := d\psi_{\psi^{-1}(q)} T(\psi^{-1}(q)),$$

for $q \in V$. Prove that $[\psi_*(X), \psi_*(Y)] = \psi_*([X, Y])$.
[*Hint:* Use that for all smooth $f\colon V \to \mathbb{R}$, and vector field T on U we have $(\psi_*(T)f) \circ \psi = T(f \circ \psi)$.]

(d) Show that if $M \subset \mathbb{R}^3$ is an embedded surface, and X, Y are tangent vector fields on M, then the Lie bracket $[X, Y]$ is well-defined as $[\tilde{X}, \tilde{Y}]$, where $\tilde{X}$, $\tilde{Y}$ are extensions of X, Y to an open set $U \subset \mathbb{R}^3$ containing M. Prove that $[X, Y]$ is also tangent to M.
[*Hint:* You may assume that such extensions always exists. Also, if $\psi\colon W \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^3$, then T is tangent to $M \cap W$ if and only if $\psi_*(T)$ is tangent to $\psi(M \cap W) \subset V$.]

(e) With M as above, show that if X, Y are tangent vector fields such that $X(p)$, $Y(p)$ is an orthonormal basis of TM_p for all $p \in M$, then

$$\langle D_X D_Y X - D_Y D_X X - D_{[X,Y]} X, Y \rangle = -K,$$

where D denotes the covariant derivative and K is the Gauss curvature.
[*Hint:* Denoting ν the unit normal to M, recall that

$$K = \langle XX, \nu \rangle \langle YY, \nu \rangle - \langle YX, \nu \rangle \langle XY, \nu \rangle.$$

Using the previous expression for K, and that X, Y, ν are orthonormal, prove $-K = \langle Y(\langle XX, \nu \rangle \nu) - X(\langle YX, \nu \rangle \nu), Y \rangle$.

Also, recall the definition of covariant derivative $D_Z T = ZT - \langle ZT, \nu \rangle \nu$ for any tangent vector fields Z, T.]

Solution: a) We have (using the summation over repeated indices convention)

$$\begin{aligned} Zf &= X^i (Y^j f_j)_i - Y^i (X^j f_j)_i = X^i Y^j_i f_j + X^i Y^j f_{ji} - Y^i X^j_i f_j - Y^i X^j f_{ji} \\ &= X^i Y^j_i f_j - Y^i X^j_i f_j = (X^i Y^j_i - Y^i X^j_i) f_j \end{aligned}$$

Hence Z is a vector field with components $Z^i = \sum_{j=1}^n (X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial}{\partial x^j} X^i)$.

b) Using the computation of $[X, Y]^i$ from a) we have

$$(XY - YX)^i = XY^i - YX^i = X^j Y^i_j - Y^j X^i_j = [X, Y]^i.$$

c) For all $f \in C^\infty(V)$ we have

$$\begin{aligned}(\psi_*([X,Y])f)\circ\psi &= [X,Y](f\circ\psi) = X(Y(f\circ\psi)) - Y(X(f\circ\psi))\\ &= X((\psi_*(Y)f)\circ\psi) - Y((\psi_*(X)f)\circ\psi)\\ &= \psi_*(X)((\psi_*(Y)f))\circ\psi - Y((\psi_*(X)f)\circ\psi)\\ &= ([\psi_*(X),\psi_*(Y)]f)\circ\psi.\end{aligned}$$

Hence $\psi_*([X,Y]) = [\psi_*(X),\psi_*(Y)]$.

d) Let be X, Y to tangent vector fields to M, and let $\tilde{X}$, $\tilde{Y}$ be extensions in an open set $U \subset \mathbb{R}^3$ which contains M.

Assume first that M is of the form $V \cap \{x^3 = 0\}$, where $V \subset \mathbb{R}^3$ is an open set. Then, given any two smooth vector fields Z, T on M with extensions $\tilde{Z}$, $\tilde{T}$ to V we have

$$[\tilde{Z},\tilde{T}]^i = \sum_{j=1}^{3}(\tilde{Z}^j \tfrac{\partial}{\partial x^j}\tilde{T}^i - \tilde{T}^j \tfrac{\partial}{\partial x^j}\tilde{Z}^i) = \sum_{j=1}^{2}(Z^j \tfrac{\partial}{\partial x^j}T^i - T^j \tfrac{\partial}{\partial x^j}Z^i)$$

on $V \cap \{x^3 = 0\}$, where we used $\tilde{Z}^j = \tilde{T}^j \equiv 0$ for $j = 3$. This also shows that $[\tilde{Z},\tilde{T}]^3 = 0$ since $\frac{\partial}{\partial x^j}Z^3 = \frac{\partial}{\partial x^j}T^3 \equiv 0$ for $j = 1, 2$. Hence $[\tilde{Z},\tilde{T}]$ is tangent and depens only on Z, T.

For general $M \subset U \subset \mathbb{R}^3$ we can use a submanifold chart $\psi : U \to V$ such that $\psi(M) = V \cap \{x^3 = 0\}$, c) and the hint. Indeed, since by c) the Lie bracket commutes with the push forward of vector fields, if $\tilde{X}, \tilde{Y}$ are extensions to U of two tangent vector fields X, Y to M, and $Z = \psi_* X$, $T = \psi_* Y$ we obtain

$$\psi_*([\tilde{X},\tilde{Y}])\circ\psi = [\psi_*(\tilde{X}),\psi_*(\tilde{Y})]\circ\psi = [\tilde{Z},\tilde{T}]\circ\psi = [Z,T]\circ\psi \text{ on } M.$$

This shows that $[\tilde{X},\tilde{Y}]$ does not depend on the chosen extensions. Using the hint we obtain that it is tangent to M.

e) We have

$$\begin{aligned}&\langle D_X D_Y X - D_Y D_X X - D_{[X,Y]}X, Y\rangle = \langle X D_Y X - Y D_X X - [X,Y]X, Y\rangle\\ &\quad= \big\langle X(YX - (\langle YX,\nu\rangle\nu)) - Y(XX - Y(\langle XX,\nu\rangle\nu)) - XYX - YXX, Y\big\rangle\\ &\quad= \big\langle XYX - X(\langle YX,\nu\rangle\nu) - YXX + Y(\langle XX,\nu\rangle\nu) - XYX - YXX, Y\big\rangle\\ &\quad= \big\langle Y(\langle XX,\nu\rangle\nu) - X(\langle YX,\nu\rangle\nu), Y\big\rangle\end{aligned}$$

Now since $\big\langle\langle XX,\nu\rangle\nu, Y\big\rangle \equiv 0$ differentiating we obtain

$$\big\langle\langle XX,\nu\rangle\nu, YY\big\rangle + \big\langle Y(\langle XX,\nu\rangle\nu), Y\big\rangle = 0$$

Similarly,

$$\big\langle\langle YX,\nu\rangle\nu, XY\big\rangle + \big\langle X(\langle YX,\nu\rangle\nu), Y\big\rangle = 0$$

Therefore,

$$\big\langle Y(\langle XX,\nu\rangle\nu) - X(\langle YX,\nu\rangle\nu), Y\big\rangle = -\langle XX,\nu\rangle\langle YY,\nu\rangle + \langle YX,\nu\rangle\langle XY,\nu\rangle = -K.$$

3. Sard's Lemma and Whitney's Embedding Theorem

Let M be a compact m-dimensional C^∞ manifold. Recall that there exists an embedding $F\colon M \to \mathbb{R}^n$ for a (possibly very large) n depending on M (Theorem 8.9 in the lecture). The goal of this problem is to lower the dimension n to $2m+1$.

(a) Let $\tilde{M} \subset \mathbb{R}^n$ be a compact m-dimensional C^∞ submanifold. Prove that

$$\mathcal{U}T\tilde{M} := \big\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \ : x \in \tilde{M}, \xi \in T\tilde{M}_x, |\xi| = 1\big\}$$

is a $(2m-1)$-dimensional compact C^∞ submanifold of $\mathbb{R}^{2n}$.
[*Hint:* Using a submanifold chart (ψ, U) notice that $\tilde{M} \cap U$ can be written as $\{x \in U : \psi^{m+1}(x) = \cdots = \psi^n(x) = 0\}$. Try to write $\mathcal{U}T\tilde{M}$ locally as the zero set of a certain map $G\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n-2m+1}$ having 0 as a regular value.]

(b) Given $e \in \mathbb{S}^{n-1}$ define

$$e^\perp := \{x \in \mathbb{R}^n : \ e \cdot x = 0\} \cong \mathbb{R}^{n-1},$$

and let $\pi_e\colon \mathbb{R}^n \to e^\perp$ be the orthogonal projection $x \mapsto x - (e \cdot x)e$. Prove that $\pi_e|_{\tilde{M}}$ is an immersion if and only if e does *not* belong to the image of the map $\pi_2|_{\mathcal{U}T\tilde{M}}\colon \mathcal{U}T\tilde{M} \to \mathbb{S}^{n-1}$, defined as the restriction of the canonical projection $\pi_2(x,\xi) = \xi$.

(c) Prove that $\pi_e|_{\tilde{M}}$ is injective if and only if $\pm e$ do *not* belong to the image of the map $g\colon (\tilde{M} \times \tilde{M}) \setminus \Delta \to \mathbb{S}^{n-1}$, defined as

$$g(x,y) := \frac{x-y}{|x-y|},$$

where $\Delta := \{(x,x)\colon x \in \tilde{M}\}$.

(d) Using Sard's Lemma, show that if $n > 2m+1$, then for almost every $e \in \mathbb{S}^{n-1}$ the projection $\pi_e\colon \tilde{M} \to e^\perp$ is an injective immersion.
[*Hint:* Recall Sard's Lemma. If $F\colon M^m \to N^n$ is a C^r map with $r > \max\{0, n-m\}$, then the set of singular values of F has measure zero in N.]

(e) Prove Whitney's Embedding Theorem (compact case), namely, that every smooth compact m-dimensional manifold can be embedded in $\mathbb{R}^{2m+1}$.

Solution: a) Using a submanifold chart $\psi : U \to V$, we have that $\tilde{M} \cap U = \{\psi^{m+1} = \cdots = \psi^n = 0\}$. Now, $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ belong to $(\mathcal{U}T\tilde{M}) \cap U \times \mathbb{R}^n$ if and only if $x \in \tilde{M} \cap U$, $\xi \in T_x\tilde{M}$, and $|\xi| = 1$. Note that $\xi \in T_x\tilde{M}$ if and only if $d\psi_x(\xi)$ is a vector with vanishing components $m+1, m+2, \ldots, n$. In other words,

$$\xi \in T_x\tilde{M} \quad \Leftrightarrow \quad \big(d\psi_x(\xi)\big)^i = \sum_{j=1}^{n} \tfrac{\partial \psi^i}{\partial x^j}(x)\xi^j = 0 \quad \text{for all } i = m+1, \ldots, n.$$

Hence $(\mathcal{U}T\tilde{M}) \cap U \times \mathbb{R}^n$ is the zero set of the function $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2m+1}$ below:

$$G(x, \xi) = (\psi^{m+1}(x), \ldots, \psi^n(x), \textstyle\sum_{j=1}^n \frac{\partial \psi^{m+1}}{\partial x^j}(x)\xi^j, \ldots, \sum_{j=1}^n \frac{\partial \psi^n}{\partial x^j}(x)\xi^j, |\xi|^2 - 1).$$

Hence, to show that $(\mathcal{U}T\tilde{M}) \cap (U \times \mathbb{R}^n)$ is a smooth $2m-1$ manifold it suffices to show that 0 is a regular value of $G|_{U \times \{0<|\xi|<2\}}$. Let us compute the matrix dG at any point $(x, \xi) \in U \times \{0 < |\xi| < 2\}$ to see that it has maximal rank. We have:

$$\tfrac{\partial}{\partial x^k}\big(\textstyle\sum_{j=1}^n \frac{\partial \psi^\ell}{\partial x^j}(x)\xi^j\big) = \sum_{j=1}^n \frac{\partial^2 \psi^\ell}{\partial x^j \partial x^k}(x)\xi^j$$

$$\tfrac{\partial}{\partial \xi^k}\big(\textstyle\sum_{j=1}^n \frac{\partial \psi^\ell}{\partial x^j}(x)\xi^j\big) = \frac{\partial \psi^\ell}{\partial x^k}(x).$$

Hence for all $(x, \xi) \in U \times \{0 < |\xi| < 2\}$, for $A^t_k(x) := \frac{\partial \psi^{m+t}}{\partial x^k}$, $B^t_k(x, \xi) := \sum_{j=1}^n \frac{\partial^2 \psi^{m+t}}{\partial x^j \partial x^k}(x)\xi^j$

$$dG_{(x,\xi)} := \begin{pmatrix} A(x) & 0 \\ B(x,\xi) & A(x) \\ 0 & 2p \end{pmatrix}.$$

Since ψ is a submanifold chart A has maximal rank, and so it is clear from its blocks structure that $dG_{(x,\xi)}$ has maximal rank too.

b) Since $\pi_e : \mathbb{R}^n \to e^\perp$ is linear the differential of its restriction $d(\pi_e|_{\tilde{M}})_x$ at $x \in \tilde{M}$ is $\pi_e(\xi) = \xi - (e \cdot \xi)e$. Here $\xi \in T\tilde{M}_x$ is any tangent vector. Hence $d(\pi_e|_{\tilde{M}})_x$ is injective if $\xi \mapsto \xi - (e \cdot \xi)e$ is injective for all $\xi \in T\tilde{M}_x$. This happens if and only if $e \notin T\tilde{M}_x$. Hence, $\pi_e|_{\tilde{M}}$ is an immersion if an only if for all $x \in \tilde{M}$, $e \notin T\tilde{M}_x$ or equivalently if $\pi_2(\mathcal{U}TM) \cap \{e\} = \{\xi \ : \exists x,\ \xi \in T\tilde{M}_x\} \cap \{e\} = \varnothing$.

c) $\pi_e|_{\tilde{M}}$ is *not* injective if and only if $\pi_e(x) = \pi_e(y)$ for different point x, y in $\tilde{M}$ (equivalently $(x, y) \in \tilde{M} \times \tilde{M} \setminus \Delta$). In other words, if and only if

$$x - (e \cdot x)e = y - (e \cdot y)e \quad \Leftrightarrow \quad x - y \in \mathbb{R}e \quad \Leftrightarrow \quad g(x, y) = \frac{x - y}{|x - y|} = \pm e.$$

d) Suppose that $n > 2m + 1$. Now since $\pi_2|_{\mathcal{U}T\tilde{M}}$ is a smooth map from a $(2m - 1)$ dimensional manifold to $\mathbb{S}^{n-1}$. Then, by Sard's lemma every $y \in \mathbb{S}^{n-1} \setminus \Sigma_1$, where Σ_1 is a set of measure zero, is a regular value of $\pi_2|_{\mathcal{U}T\tilde{M}}$. Since $n - 1 > 2m - 1$ this is only possible if y does not belong to the image of this map. Hence, using b), $\pi_e|_{\tilde{M}}$ is an immersion for all $e \in \mathbb{S}^{n-1} \setminus \Sigma_1$. Similarly, since $g|_{M \times \tilde{M} \setminus \Delta}$ is a smooth map from a $(2m)$-dimensional manifold to $\mathbb{S}^{n-1}$. Then, by Sard's lemma every $y \in \mathbb{S}^{n-1} \setminus \Sigma_2$, where Σ_2 is a set of measure zero, is a regular value of $g|_{M \times \tilde{M} \setminus \Delta}$. Since $n - 1 > 2m$ this is only possible if y does not belong to the image of this map. Hence, using c), $\pi_e|_{\tilde{M}}$ is an immersion for all $e \in \mathbb{S}^{n-1} \setminus (\Sigma_2 \cup -\Sigma_2)$. So we have shown that for all $e \in \mathbb{S}^{n-1}$ not belonging to the null set $\Sigma_1 \cup \Sigma_2 \cup -\Sigma_2$ the map $\pi_e|_{\tilde{M}}$ is a injective immersion of the compact manifold $\tilde{M}$ in $e^{\perp} \cong \mathbb{R}^{n-1}$, hence an embedding. Repeating this procedure we find that successive projections of $\tilde{M}$ are embeddings in lower dimensional spaces, until we reach the critical dimension $2m + 1$ in which the previous argument based on Sard's lemma does not work anymore.

Differential Geometry I exam (multiple choice part)

1. The radius of the osculating circle of the curve $c(t) := (at, -t^2)$, $a > 0$, at the point $(0, 0)$ is given by:

√ (a) $\frac{a^2}{2}$.

(b) $\frac{a}{2}$.

(c) 2.

(d) 1.

(e) $\frac{\sqrt{a}}{2}$.

Solution. Consider the re-parametrization $s \mapsto (s, -(s/a)^2)$. At the point $(0, 0)$ it has unit speed and acceleration with modulus $2/a^2$ and perpendicular to the velocity. Hence, the curvature at $(0, 0)$ is given by $2/a^2$. Therefore the radius of curvature is $a^2/2$

2. Assume that a (smooth, nonempty) compact 2-dimensional submanifold $M \subset \mathbb{R}^3$ satisfies $\int_M H^2\, dA = \int_M K\, dA$, where K is the Gauss curvature and H is the mean curvature[1]. Then M must be a/an

(a) sphere.

√ (b) union of spheres.

(c) ellipsoid.

(d) Clifford torus.

(e) point.

Solution.

If k_1, k_2 denote the principal curvatures of M, then $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$. We compute

$$4H^2 = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1k_2 = (k_1 - k_2)^2 + 4K$$

Hence $\int_M H^2 = \int_M K$ forces $k_1 = k_2$ at every point. That is every point must be umbilical. Happens if and only if (see Theorem 4.6 in the lecture) every connected component of our (compact!) surface must is a sphere.

[1] The average of the principal curvatures.

3. Consider the differential 1-form $\omega = -ydx + xdy$ in $\mathbb{R}^2$. Let D be the ellipsoid $\{(x,y) : ax^2 + \frac{1}{a}y^2 \leq 1\}$, where $a > 0$. Then $\int_{\partial D} \omega$ equals:

√ (a) 2π.

(b) $\pi\big(a + \frac{1}{a}\big)$.

(c) $e^{2\pi i}$.

(d) $-\pi\cos(\mathrm{a})$.

(e) 1.

Solution. By Stokes' theorem $\int_{\partial D} \omega = \int_D d\omega = \int_D 2dxdy$, that is 2 times the area of D. Since for all $b > 0$ the map $(x,y) \to (bx, y/b)$ is area preserving (the Jacobian equals 1) the are of D equals the area of the unit disc. That is π. Hence the correct answer is 2π.

4. Let $C \subset \mathbb{R}^3$ be the cylinder in $\mathbb{R}^3$, parametrized as

$$f(u,v) = \big(R\cos u, R\sin u, v^3\big),$$

$R > 0$. What are the correct values of the Gauss curvature K and mean curvature H at the point $(R, 0, \sqrt[3]{2}) \in C$ (with respect to the outward pointing Gauss map)?

(a) $K = 0, H = 0$.

√ (b) $K = 0, H = -\frac{1}{2R}$.

(c) $K = 0, H = \frac{1}{2R} + 1$.

(d) $K = \frac{2}{R^2}, H = \frac{1}{R}$.

(e) $K = 0, H = -R/2$.

Solution Being a cylinder of radius R, at any point the principal curvatures are $k_1 = -1/R$ and $k_2 = 0$. Thus $K = 0$ and $H = -\frac{1}{2R}$.

5. Consider a "spherical pentagon" (geodesically convex region bounded by five circular arcs) of area A in a 2-sphere of $\mathbb{R}^3$ with area A'. The sum of its (five) interior angles is:

√ (a) $\pi\big(3 + 4\frac{A}{A'}\big)$.

(b) $2\pi + A/A'$.

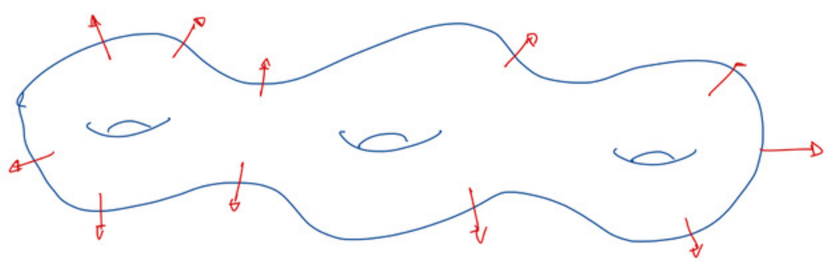
(c) $3\pi + A$.

(d) $5\pi + 2A/\sqrt{A'}$.

(e) $5\pi + A/\sqrt{A'}$

Solution As proven in the first introductory lecture (or as a consequence of Gauss's "theorema elegantissimum"), the sum of the angles of a spherical triangle of area $\widetilde{A}$ in a sphere or area A' of radius r is $\pi(1 - 4\widetilde{A}/A')$. Decompose the pentagon as union of three triangles.

6. Let $M \subset \mathbb{R}^3$ be the smooth surface as depicted:



What is the value of the integral of the Gauss curvature K over M (with respect to the differential of the area)?

(a) -3π.

(b) 0.

(c) depends on how M is embedded in $\mathbb{R}^3$.

(d) -6π.

√ (e) -8π.

Solution The sketch shows a genus three surface, so its Euler characteristic is 2(1-3) = -4. Hence, by Gauss-Bonnet $\int_M K dA = 2\pi\chi(M) = -8\pi$.

7. Which one is true?

(a) Cylinders, spheres and planes are the only connected submanifolds of $\mathbb{R}^3$ with constant mean curvature.

√ (b) A smooth compact surface in $\mathbb{R}^3$ whose area is minimal among all surfaces enclosing the same volume must have constant mean curvature.

(c) Any connected embedded minimal surface $M \subset \mathbb{R}^3$ with $\int_M K \, dA > -8\pi$ must be a plane.

(d) Alexandroff's theorem classifies all constant mean curvature embedded surfaces.

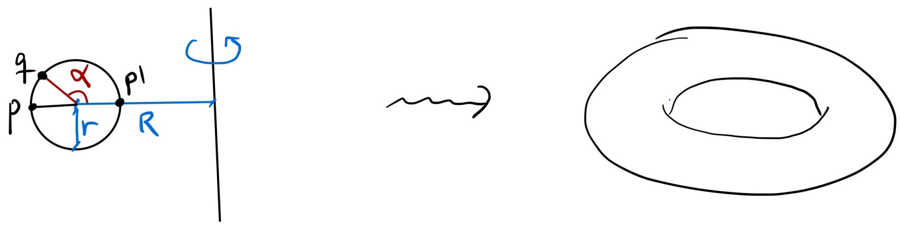
(e) If a constant mean curvature surface is embedded and non-compact, then it must be a cylinder.

Solution As we saw in the lecture, any smooth surface with minimal area among all those enclosing the same (finite) volume must have constant mean curvature. The other statements are all false, the catenoid is a counterexample to (a), (c), (d), (e).

8. Consider the torus of revolution

$$f(x,y) = \big(\cos x(-R + r\cos y), \sin x(-R + r\cos y), r\sin y\big),$$

$R > r > 0$, drawn below:



Its mean curvature (with respect to the outward pointing Gauss map) at $p = (-R - r, 0, 0)$ is:

(a) $-\frac{1}{2}\big(\frac{1}{r} + \frac{1}{\sqrt{R^2-r^2}}\big)$.

(b) $-\frac{1}{2}\big(\frac{1}{\sqrt{rR}} + \frac{1}{R-r}\big)$.

✓ (c) $-\frac{1}{2}\big(\frac{1}{r} + \frac{1}{R+r}\big)$.

(d) $-\frac{1}{2}\big(\frac{1}{r} - \frac{1}{R+r}\big)$.

(e) $-\frac{1}{2}\big(\frac{1}{r} - \frac{1}{\sqrt{R^2-r^2}}\big)$.

Solution At the point p both principal curvatures have the same sign. One corresponds to the meridian and equals $-\frac{1}{r}$ (the meridian has radius r). The other corresponds to the parallel through p, which is a geodesic (by symmetry) traces a circumference of radius $R+r$, so the curvature gives $-\frac{1}{R+r}$. Then mean curvature is its average of the two principal curvatures computed before.

9. Consider again the torus from the previous question. At any point of the torus one principal curvature is $-1/r$. The other principal curvature at the point $q = (-R + r\cos\alpha, 0, r\sin\alpha)$ is:

√ (a) $\frac{\cos\alpha}{R - r\cos\alpha}$.

(b) $\frac{\cos\alpha}{R-r}$.

(c) $\frac{\tan\alpha}{R-r}$.

(d) $\frac{\cos\alpha}{R - r\sin\alpha}$.

(e) $\frac{\tan\alpha}{R+r}$.

Solution The principal curvature $-\frac{1}{r}$ corresponds to the meridians. To compute the principal curvature in the orthogonal direction at q we consider the circumference $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$. Its curvature is $1/(R - r\cos\alpha)$, so after projecting (the normal vector to the circumference at q points in the (1,0) while the normal to the surface is $(\cos\alpha, \sin\alpha)$ at q) we obtain that the normal curvature is $\frac{\cos\alpha}{R - r\cos\alpha}$ direction. Since γ is a curve trough q contained in the surface and velocity vector orthogonal to the other principal direction, its normal curvature is precisely the principal curvature we are trying to compute.

10. Consider again the torus from the previous two questions. When the point q is rotated about the x_3-axis, it generates the curve $\gamma(t) = (\cos t(-R + r\cos\alpha), \sin t(-R + r\cos\alpha), r\sin\alpha)$, which is contained in the torus. Given a tangent vector X at q consider its parallel transport along γ for one full turn ($t \in [0, 2\pi]$), producing a new tangent vector Y at q. The angle between X and Y is:

(a) $\frac{\alpha R}{r}$.

√ (b) $2\pi\sin\alpha$.

(c) $\frac{\tan\alpha R}{r}$.

(d) $2\pi\cos\alpha$.

(e) $\sin\alpha$.

Solution Consider the cone tangent to the torus along γ. It is a cone of revolution (also with respect the x_3 axis) and the angle of its generating lines of the cone and the x_3 axis is $\pi - \alpha$. Hence when "opening" the cone (as we saw in the lecture in the it becomes a flat example of Foucault's pendulum) it becomes a flat circular sector of angle $2\pi\sin\alpha$. Hence, since parallel transport is trivial for the flat surface, we see that the angle between the transported vector and the original one is $2\pi\sin\alpha$.