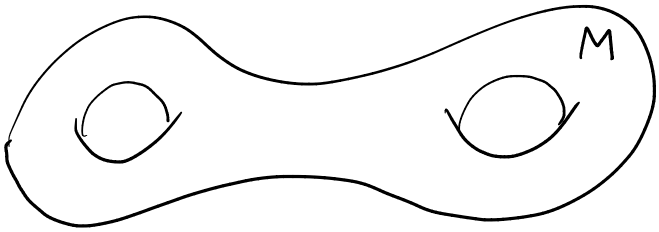
Solutions Differential Geometry I

Single Choice Questions

SC1 Let M be the (genus 2) surface in the figure.



Then $\int_M K\,dA$ equals:

(A) 4π.

(B) 0.

(C) -2π.

(D) -4π.

(E) -8π.

Solution:

The correct solution is (d). The surface has genus $g = 2$, so $\chi(M) = 2 - 2g = -2$ and by Gauss-Bonnet $\int_M K\,dA = 2\pi\chi(M) = -4\pi$.

SC2 Let M be the same surface as in question SC1. If $N\colon M \longrightarrow \mathbb{S}^2$ is the Gauss map, then $\deg_2 N$, i.e. the mapping degree mod 2, equals

(A) 0.

(B) 1.

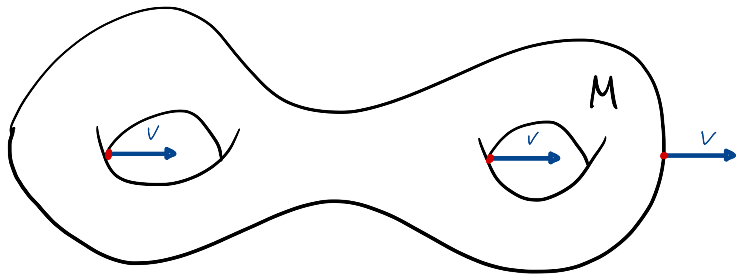
(C) depends on whether N is pointing inwards or outwards.

(D) 1 at points of positive Gauss curvature, 0 otherwise.

(E) 0 at points of positive Gauss curvature, 1 otherwise.

Solution:

The correct solution is (b). Consider the outward pointing Gauss map and let v be unit normal vector pointing to the "right". Then $N^{-1}(v)$ consists of the three points p on M where $N(p) = v$, as shown in picture. They are all regular points (and hence v is a regular value) because the Gauss curvature does not vanish. Thus $\deg_2 N = \# N^{-1}(v) \bmod 2 = 1$.



SC3 Consider the following (graphical) submanifolds of $\mathbb{R}^3$, parametrized by $f\colon (x, y) \mapsto (x, y, z(x, y))$, where $z = z(x, y)$ is given by

(I) $z = x^2 + y^2$ (II) $z = x^2 - y^2$ (III) $z = x^4 - y^2$

(IV) $z = x^2 - \sin^2 y$ (V) $z = y^2 + (x + y)^3$

The following is true.

(A) (I) and (V) have positive Gauss curvature at $(0, 0)$.

(B) (II), (III), and (IV) have negative Gauss curvature at $(0, 0)$.

(C) (III) and (V) have zero Gauss curvature at $(0, 0)$.

(D) (II) is the only one with negative Gauss curvature at $(0, 0)$.

(E) (V) has positive Gauss curvature at $(0, 0)$.

Solution:

The correct solution is (c). Geometrically, or with a direct computation one shows that the sign of the Gauss curvatures at $(0, 0)$ are:

$$K_{\mathrm{I}} > 0, \qquad K_{\mathrm{II}} < 0, \qquad K_{\mathrm{III}} = 0, \qquad K_{\mathrm{IV}} < 0, \qquad K_{\mathrm{V}} = 0.$$

SC4 Suppose $\Omega \subset \mathbb{R}^3$ is an open subset such that $\partial\Omega$ (the topological boundary) is a smooth compact submanifold satisfying $\int_{\partial\Omega} |K|\, dA = 4\pi$. Then

(A) Ω must be convex.

(B) Ω must be a ball.

(C) Ω must be a union of disjoint balls.

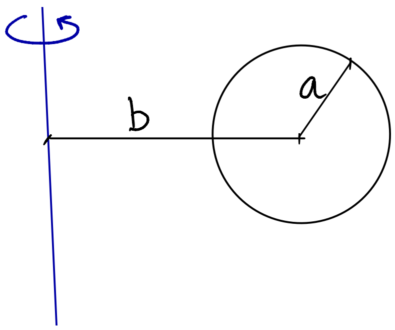
(D) Ω must have constant mean curvature.

(E) the mean curvature of Ω must vanish at one point, at least.

Solution:

The correct solution is (a). For the sphere $\mathbb{S}^2$ we have $\int_{\mathbb{S}^2} |K|\, dA = 4\pi$, so (c) and (e) are wrong. Attaching two hemispheres to a cylinder we obtain a surface M with $\int_M |K|\, dA = \int_{\mathbb{S}^2} |K|\, dA = 4\pi$ (because the Gauss curvature K vanishes along the cylindrical portion of M), thus (b) and (d) are wrong.

SC5 Consider the torus of revolution as in the figure.



Let Ω be the set of points with positive Gauss curvature. The area of Ω is:

(A) $2\pi^2 ab$.

(B) $2\pi^2 ab + 4\pi a^2$.

(C) $4\pi ab$.

(D) $2\pi(a^2 + b^2)$.

(E) $\pi^2 a^2 + \pi ab$.

Solution:

The correct solution is (b). An explicit parametrization of the torus is given by $f\colon (-\pi, \pi) \times (0, 2\pi) \longrightarrow \mathbb{R}^3$,

$$f(x, y) = \Big((b + a\cos x)\cos y, (b + a\cos x)\sin y, a\sin x\Big).$$

Then one computes

$$g(x, y) = \begin{pmatrix} a^2 & 0 \\ 0 & (b + a\cos x)^2 \end{pmatrix}, \qquad K(x, y) = \frac{\cos x}{a(b + a\cos x)}.$$

Thus $\Omega = f(\{-\pi/2 < x < \pi/2\})$ and

$$\mathrm{A}(\Omega) = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} a(b + a\cos x)\, dxdy = 2\pi^2 ab + 4\pi a^2.$$

SC6 For Ω as in question SC5, the integral $\int_\Omega K\, dA$ equals:

(A) 4π.

(B) $\pi^2 \dfrac{b-a}{a+b}$.

(C) $\dfrac{8\pi ba}{a^2+b^2}$.

(D) $2\pi\Big(1 + \dfrac{2ab}{a^2+b^2}\Big)$.

(E) $4\pi \dfrac{a^2}{a^2+b^2}$.

Solution:

The correct solution is (a). From question 5 we get

$$\int_\Omega K\, dA = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos x}{a(b + a\cos x)} a(b + a\cos x)\, dxdy = 4\pi.$$

Problem 1
The sphere revisited

Consider the parametrization f given by the inverse of the stereographic projection $\mathbb{S}^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$ from the North pole. Explicitly, $f\colon \mathbb{R}^2 \longrightarrow \mathbb{S}^2 \setminus \{(0,0,1)\}$ is given by

$$f\colon (x,y) \longmapsto \Big(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\Big).$$

(a) Compute the first fundamental form of f. Is f conformal?

Solution:

We have

$$f_x = \left(\frac{2-2x^2+2y^2}{(1+x^2+y^2)^2}, \frac{-4xy}{(1+x^2+y^2)^2}, \frac{4x}{(1+x^2+y^2)^2}\right).$$

$$f_y = \left(\frac{-4xy}{(1+x^2+y^2)^2}, \frac{2+2x^2-2y^2}{(1+x^2+y^2)^2}, \frac{4y}{(1+x^2+y^2)^2}\right).$$

Then we can compute

$$g_{11} = \langle f_x, f_x \rangle = \frac{4}{(1+x^2+y^2)^2}$$

Also $g_{22} = g_{11}$ (since g_{11} is symmetric in x, y). Moreover, a direct computation shows $g_{12} = g_{21} = \langle f_x, f_y \rangle = 0$. So

$$g(x,y) = \frac{4}{(1+x^2+y^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and f is conformal.

Consider $\mathbb{R}^2$, the domain of the parametrization f. The *metric distance* between two points (x, y) and (x', y') is defined as

$$\inf\left\{L(\gamma)\,\middle|\,\gamma\colon[0,1]\to\mathbb{R}^2 \text{ piecewise smooth curve with } \gamma(0)=(x,y), \gamma(1)=(x',y')\right\},$$

where the length L of the curve γ is defined as $L(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt$, and g is the first fundamental form of the parametrization f.

(b) Show that for any parallel, represented on $\mathbb{R}^2$ by a circle $x^2 + y^2 = r^2$ for some $r > 0$, the *metric distance* between $(0, 0)$ and any of its points (x_0, y_0), depends only on r and is attained by the curve $s \longmapsto s(x_0, y_0)$, defined for $s \in [0, 1]$.

Hint: For $a \in \mathbb{R}$, $\int \frac{1}{a^2+s^2}\,ds = \frac{1}{a}\arctan(\frac{s}{a}) + c$.

Solution:

Fix (x_0, y_0) on the circle $C_r := \{x^2 + y^2 = r^2\}$ and let $\gamma : [0, 1] \longrightarrow \mathbb{R}^2$ be any piecewise C^1 curve joining $(0, 0)$ and (x_0, y_0). Let $t' = \min\{t > 0 \mid \gamma(t) \in C_r\}$ and let $(x', y') = \gamma(t')$. Let us show that $L(\gamma|_{[0,t']}) \geq L(\tilde{\gamma})$ where $\tilde{\gamma}(s) = s(x', y')$, defined for $s \in [0, 1]$.

Indeed,

$$\begin{aligned} L(\gamma|_{[0,t']}) &= \int_0^{t'} \sqrt{g_{\gamma(t)}(\gamma'(t),\gamma'(t))}dt = 2\int_0^{t'} \frac{|\gamma'(t)|}{1+\gamma(t)^2}dt \geq 2\int_0^{t'} \frac{|\gamma(t)|'}{1+\gamma(t)^2}dt \\ &= 2\Big[\arctan|\gamma(t)|\Big]_0^{t'} = 2\Big[\arctan|\tilde{\gamma}(t)|\Big]_0^1 = 2\int_0^1 \frac{ds}{1+s^2} = L(\tilde{\gamma}). \end{aligned}$$

Finally we conclude noticing that $L(\gamma) \geq L(\gamma|_{[0,t']})$ and that, by rotational symmetry of g, $L(\tilde{\gamma})$ equals the length of $s \mapsto s(x_0, y_0)$.

(c) Prove that the distance between any two points on the sphere (which are not antipodal) is attained by the great circular arc joining them.

Hint: you can either deduce it using (b) or give another proof, e.g. the one that was given in the first lecture.

Solution:

Since $SO(2)$ is a group of isometries acting transitively on $\mathbb{S}^2$ (i.e. given any two points, there exists at least one isometry sending the first point to the second) we can assume without loss of generality that one of the two points is the "south pole" $\boldsymbol{S} := (0, 0, -1)$. If $q \neq \boldsymbol{N} := (0, 0, 1)$ then $q = f(x_\circ, y_\circ)$ for some $(x_\circ, y_\circ) \in \mathbb{R}^2$. Then given any curve $c : [0, 1] \longrightarrow \mathbb{S}^2 \setminus \boldsymbol{N}$ joining S and $q \in f(\{x^2 + y^2 = r^2\})$ can be written as $\tilde{c} = f \circ \gamma$, for some $\gamma : [0, t_\circ] \longrightarrow \mathbb{R}^2$. By c). the metric distance between $(0, 0)$ and $(x_\circ, y_\circ)$ is attained by the ray $t \mapsto t(x_\circ, y_\circ)$, (notice by definition of length in local coordinates $L(\gamma) = L(c)$). Finally, any ray through the origin is mapped to a great circular arc by f.

Problem 2

Local isothermal coordinates on a minimal surface

(a) Suppose that U is a ball in the (x, y)-plane $\mathbb{R}^2$. Show that, given smooth functions $A, B\colon U \longrightarrow \mathbb{R}$, the equations $\frac{\partial}{\partial x}\Phi = A$, $\frac{\partial}{\partial y}\Phi = B$ admit a solution Φ if and only if $\frac{\partial}{\partial y}A = \frac{\partial}{\partial x}B$ holds.

Solution:

Let $(x_\circ, y_\circ)$ be the center of U. Define $\Phi(x, y) := \int_{x_\circ}^x A(t, y_\circ)dt + \int_{y_\circ}^y B(x, t)dt + C$. Then

$$\Phi_x = A(x, y_\circ) + \int_{y_\circ}^y B_x(x, t)dt = A(x, y_\circ) + \int_{y_\circ}^y A_y(x, t)dt = A(x, y)$$

and $\Phi_y = B(x, y)$ as desired.

(b) Let $U \subset \mathbb{R}^2$ be a ball and let $v\colon U \longrightarrow \mathbb{R}$ be such that $f\colon U \longrightarrow \mathbb{R}^3$,

$$f(x, y) := (x, y, v(x, y))$$

is a minimal immersion (i.e. the mean curvature or the trace of the Weingarten map vanish for all (x, y)). Show that v must satisfy the PDE

$$\frac{\partial}{\partial x}\left(\frac{v_x}{W}\right) + \frac{\partial}{\partial y}\left(\frac{v_y}{W}\right) = 0, \tag{$\circledast$}$$

where $W = W(x, y) := \sqrt{1 + v_x^2 + v_y^2}$, and where the subindexes x and y denote partial derivatives.

Solution:

We compute $f_x = (1, 0, v_x)$, $f_y = (0, 1, v_y)$, then the matrix of the first fundamental form is

$$(g_{ij}) = \begin{pmatrix} 1 + v_x^2 & v_x v_y \\ v_x v_y & 1 + v_y^2 \end{pmatrix}$$

with $\det g = W^2$ and

$$(g^{ij}) = (g_{ij})^{-1} = \frac{1}{W^2}\begin{pmatrix} 1 + v_y^2 & -v_x v_y \\ -v_x v_y & 1 + v_x^2 \end{pmatrix}.$$

A Gauss map for f is $\nu = (-v_x, -v_y, 1)/W$, and we compute $f_{xx} = (0, 0, v_{xx})$, $f_{yy} = (0, 0, v_{yy})$ and $f_{xy} = (0, 0, v_{xy})$, so

$$\langle f_{xx}, \nu\rangle = v_{xx}/W, \qquad \langle f_{yy}, \nu\rangle = v_{yy}/W, \qquad \langle f_{xy}, \nu\rangle = v_{xy}/W,$$

Then the matrix of the second fundamental form is

$$(h_{ij}) = \frac{1}{W}\begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix}$$

and the matrix of the Weingarten map is

$$({h^i}_j) = (g^{ij})(h_{ij}) = \frac{1}{W^3}\begin{pmatrix} (1 + v_y^2)v_{xx} - v_x v_y v_{xy} & * \\ * & (1 + v_x^2)v_{yy} - v_x v_y v_{xy} \end{pmatrix}.$$

Thus f is minimal, that is, the mean curvature of f is zero if and only if $(1+v_y^2)v_{xx}-2v_xv_yv_{xy}+(1+v_x^2)v_{yy}=0$. Now a computations shows that this equation is equivalent to ($\circledast$).

(c) Show that equation ($\circledast$) from part (b) implies

$$\frac{\partial}{\partial x}\Big(\frac{1+v_y^2}{W}\Big)=\frac{\partial}{\partial y}\Big(\frac{v_xv_y}{W}\Big),$$

and deduce the existence of a potential Φ for the vector field $(\frac{v_xv_y}{W},\frac{1+v_y^2}{W})$, that is, a function $\Phi\colon U\longrightarrow\mathbb{R}$ such that

$$\frac{\partial}{\partial x}\Phi=\frac{v_xv_y}{W} \qquad\qquad \frac{\partial}{\partial y}\Phi=\frac{1+v_y^2}{W}.$$

Solution:

Since $\left(\frac{v_y}{W}\right)_y=-\left(\frac{v_x}{W}\right)_x$, we have

$$\begin{aligned}\left(\frac{v_xv_y}{W}\right)_y&=\frac{v_{xy}v_y}{W}-v_x\left(\frac{v_x}{W}\right)_x=\frac{v_{xy}v_y}{W}-\frac{v_{xx}v_x}{W}+\frac{v_x^2W_x}{W^2}\\&=\frac{v_{xy}v_y}{W}-\frac{v_{xx}v_x}{W}+\frac{(W^2-(1+v_y^2))W_x}{W^2}\end{aligned}$$

Since $W=\sqrt{1+v_x^2+v_y^2}$ we have $W_x=\frac{\partial}{\partial x}\sqrt{1+v_x^2+v_y^2}=(v_xv_{xx}+v_yv_{yx})/W$ and thus

$$-\frac{v_{xx}v_x}{W}+W_x=\frac{-v_{xx}v_x+WW_x}{W}=\frac{v_yv_{yx}}{W},$$

Therefore,

$$\left(\frac{v_xv_y}{W}\right)_y=\frac{2v_{xy}v_y}{W}-\frac{(1+y^2)(W_x)}{W^2}=\left(\frac{1+v_y^2}{W}\right)_x$$

Finally, the existence of Φ follows from **(a)**.

(d) Introduce new coordinates $\bar{x} = x$, $\bar{y} = \Phi(x,y)$ and check that the first fundamental form with respect to the new coordinates is of the form

$$\bar{g}_{(\bar{x},\bar{y})} = \lambda(\bar{x},\bar{y}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Compute the conformal factor $\lambda(\bar{x},\bar{y})$.

Hint: put $(\bar{x},\bar{y}) = \Psi(x,y) := (x, \Phi(x,y))$ and compute the first fundamental form of the parametrization $f \circ \Psi^{-1} \colon \Psi(U) \longrightarrow f(U)$, $(\bar{x},\bar{y}) \longmapsto f(\Psi^{-1}(\bar{x},\bar{y}))$, using the chain rule.

Solution:

We have

$$d\Psi_{(x,y)} = \begin{pmatrix} \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial y} \\ \frac{\partial \bar{y}}{\partial x} & \frac{\partial \bar{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \Phi_x & \Phi_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{v_x v_y}{W} & \frac{1+v_y^2}{W} \end{pmatrix}$$

Hence

$$d\Psi^{-1}_{(\bar{x},\bar{y})} = \begin{pmatrix} 1 & 0 \\ -\frac{v_x v_y}{1+v_y^2} & \frac{W}{1+v_y^2} \end{pmatrix} \left(\Psi^{-1}(\bar{x},\bar{y})\right)$$

Therefore, when computing the first fundamental form of $\bar{f} := f \circ \Psi^{-1}$ we can use

$$d\bar{f}_{(\bar{x},\bar{y})} = df_{\Psi^{-1}(\bar{x},\bar{y})} d\Psi^{-1}_{(\bar{x},\bar{y})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{v_x v_y}{1+v_y^2} & \frac{W}{1+v_y^2} \end{pmatrix} \circ \Psi^{-1}(\bar{x},\bar{y})$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{v_x v_y}{1+v_y^2} & \frac{W}{1+v_y^2} \\ \frac{v_x}{1+v_y^2} & \frac{v_y W}{1+v_y^2} \end{pmatrix} \circ \Psi^{-1}(\bar{x},\bar{y})$$

Then, the metric matrix at $(\bar{x},\bar{y})$ is given by

$$d\bar{f}^T_{(\bar{x},\bar{y})} d\bar{f}_{(\bar{x},\bar{y})} = \begin{pmatrix} 1 & -\frac{v_x v_y}{1+v_y^2} & \frac{v_x}{1+v_y^2} \\ 0 & \frac{W}{1+v_y^2} & \frac{v_y W}{1+v_y^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{v_x v_y}{1+v_y^2} & \frac{W}{1+v_y^2} \\ \frac{v_x}{1+v_y^2} & \frac{v_y W}{1+v_y^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(1+v_y^2)^2 + v_x^2 v_y^2 + v_x^2}{(1+v_y^2)^2} & 0 \\ 0 & \frac{W^2(1+v_y^2)}{(1+v_y^2)^2} \end{pmatrix} = \begin{pmatrix} \frac{W^2}{(1+v_y^2)} & 0 \\ 0 & \frac{W^2}{(1+v_y^2)} \end{pmatrix}$$

(e) Write the statement and proof of a theorem given in the lecture concerning the harmonicity of the coordinate functions for an isothermal parametrization of a minimal surface.

Solution:

Proposition 5.7 in the notes.