Mathematical Foundations for Finance

Exercise Sheet 1

The first three exercises of this sheet ask to establish some of the claims given in the lecture. The rest of the exercises contain material which is fundamental for the course and ***assumed to be known****. Please hand in your solutions by 12:00 on Wednesday, September 28 via the course homepage.*

Exercise 1.1 Recall that

$$\mathcal{G} := \{g = G_T(\vartheta) : \vartheta \in \Theta\}$$

is the set of all possible final (time-T) wealth amounts one can generate from zero initial capital, and that

$$L^0_+ := \{X : X \geqslant 0\}$$

is the set of (equivalence classes of) random variables that are (P-a.s.) nonnegative. Recall also that for two sets A and B, $A - B := \{a - b : a \in A, b \in B\}$ denotes the set of ordered differences between elements of A and B.

Prove that absence of arbitrage, i.e. $\mathcal{G} \cap L^0_+ = \{0\}$, is equivalent to

$$(\mathcal{G} - L^0_+) \cap L^0_+ = \{0\}.$$

Exercise 1.2 In the lectures, we saw that a sufficient condition for guaranteeing the absence of arbitrage is

$$E[g] = 0, \qquad \forall g \in \mathcal{G}, \tag{1}$$

i.e. that any possible wealth amount one can generate from zero initial capital has expectation zero.

(a) Prove that for a nonnegative random variable $X \in L^0_+$,

$$E[X] = 0 \qquad \Longrightarrow \qquad X = 0 \quad P\text{-a.s.}$$

(b) Using part (a), explain why (1) is indeed a sufficient condition for the absence of arbitrage.

Exercise 1.3 Consider a financial contract where the seller of the contract is obligated to pay the (random) amount H at expiry (time T) to the buyer. Define the set

$$\mathcal{Q} := \{\text{probability measure } Q \approx P : E_Q[g] \leqslant 0, \ \ \forall g \in \mathcal{G}\}.$$

Recall that the fundamental theorem of asset pricing asserts that this set is nonempty.

(a) The *superreplication price* of the contract is given by

$$\pi(H) := \inf\{x \in \mathbb{R} : \exists \vartheta \text{ with } V_T(x, \vartheta) \geqslant H \ P\text{-a.s.}\}.$$

Prove that

$$\pi(H) \geqslant \sup\{E_Q[H] : Q \in \mathcal{Q}\}.$$

(Note: it is in fact true that $\pi(H) = \sup\{E_Q[H] : Q \in \mathcal{Q}\}$, but the proof of "$\leqslant$" relies on separation theorems and is therefore beyond the scope of this exercise. This equality is known as the *hedging duality.*)

(b) For $x \in \mathbb{R}$, define the set

$$\begin{aligned}\mathcal{C}(x) &= \{\mathcal{F}_T\text{-measurable } f : \exists \vartheta \in \Theta \text{ with } V_T(x, \vartheta) \geqslant f \ P\text{-a.s.}\} \\ &= \{\mathcal{F}_T\text{-measurable } f : \exists g \in \mathcal{G} \text{ with } f \leqslant x + g\} \\ &= x + \mathcal{G} - L^0_+(\mathcal{F}_T)\end{aligned}$$

to be the collection of all possible time-T payoffs that are affordable from initial capital x via trading in the financial market.

Prove that

$$f \in \mathcal{C}(x) \qquad \Longrightarrow \qquad \sup_{Q \in \mathcal{Q}} E_Q[f] \leqslant x.$$

(Note: in fact, these two statements are equivalent, but the implication "$\Leftarrow$" is considerably more difficult.)

Exercise 1.4 Let Ω be a set.

(a) Suppose that $\{\mathcal{F}_j\}_{j \in J}$ is a nonempty family of σ-fields on Ω. Prove that the intersection $\bigcap_{j \in J} \mathcal{F}_j$ is a σ-field on Ω.

(b) Let $\mathcal{A}$ be a family of subsets of Ω. Show that there is a (clearly unique) minimal σ-field $\sigma(\mathcal{A})$ containing $\mathcal{A}$. Here minimality is with respect to inclusion: if $\mathcal{F}$ is a σ-field with $\mathcal{A} \subseteq \mathcal{F}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.

(c) Give an example of two σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega = \{1, 2, 3\}$ whose union $\mathcal{F}_1 \cup \mathcal{F}_2$ is *not* a σ-field.

Exercise 1.5 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Then the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E[X \mathbb{1}_A] = E[Z \mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called the *conditional expectation of X given $\mathcal{G}$* and is denoted by $E[X \mid \mathcal{G}]$. (This is the formal definition of conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.)

(a) Show that if X is $\mathcal{G}$-measurable, then $E[X \mid \mathcal{G}] = X$ P-a.s.

(b) Show that $E[E[X \mid \mathcal{G}]] = E[X]$.

(c) Show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$ (that is, if $\mathcal{G}$ is P-trivial), then $E[X \mid \mathcal{G}] = E[X]$ P-a.s.

(d) Consider an integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and constants $a, b \in \mathbb{R}$. Show that $E[aX + bY \mid \mathcal{G}] = aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e., there exists a collection $(A_i)_{i=1,\dots,n}$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \dots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \dots, n$. Show that

$$E[X \mid \mathcal{G}] = \sum_{i=1}^n E[X \mid A_i] \mathbb{1}_{A_i} \qquad P\text{-a.s.}$$

This says that the conditional expectation of a random variable given a finitely generated σ-algebra is a *piecewise constant* function with the constants given by the elementary conditional expectations given the sets of the generating partition.

[*This is a very useful property when one conditions on a finitely generated σ-field, as for instance in the multinomial model.*]

Hint 1: Recall that $E[X \mid A_i] = E[X \mathbb{1}_{A_i}] / P[A_i]$ and try to write X as a sum of random variables each of which only takes non-zero values on a single A_i.
Hint 2: Check that any set $A \in \mathcal{G}$ has the form $\cup_{j \in J} A_j$ for some $J \subseteq \{1, \dots, n\}$.