Mathematical Foundations for Finance

Exercise Sheet 10

Please hand in your solutions by 12:00 on Wednesday, November 30 via the course homepage.

Exercise 10.1 Let W be a Brownian motion with respect to P and $\mathbb{F}$, and for each partition Π of $[0,\infty)$, $t \geqslant 0$, and $\alpha \in [0,1]$, define

$$I_t^\alpha(W;\Pi) := \sum_{t_i \in \Pi} \Big(\alpha W_{t_i \wedge t} + (1-\alpha) W_{t_{i+1}\wedge t}\Big)\Big(W_{t_{i+1}\wedge t} - W_{t_i \wedge t}\Big).$$

For a refining sequence of partitions (Π_n) of $[0,\infty)$ with mesh decreasing to zero, find

$$\lim_{n\to\infty} I_t^\alpha(W;\Pi_n).$$

What do you get for the cases when $\alpha = 0, \frac{1}{2}, 1$?

Hint: You may use Theorem 4.1.4 from Chapter 4.

Exercise 10.2 Let $Y_1, Y_2, \dots$ be a sequence of square-integrable and independent random variables. For each $n \in \mathbb{N}_0$, set

$$X_n = \sum_{i=1}^n Y_i,$$

and let $\mathbb{F}$ be the filtration generated by X.

(a) Find What is the Doob decomposition $X = X_0 + M + A$ of X.

Simplify in the case that the Y_i have the same distribution.

(b) Compute $[M]$ and $\langle M \rangle$.

Simplify the above quantities in the case that the Y_i have the same distribution.

Hint. To find $[M]$ and $\langle M \rangle$, write out their defining properties in discrete time and interpret Δ not as "jump", but as "increment".

Exercise 10.3 Let W be a Brownian motion with respect to P and $\mathbb{F}$. Prove that for each $0 \leqslant s < t$,

$$E[W_t^3 - W_s^3 \mid \mathcal{F}_s] = 3(t-s)W_s = E\left[\int_s^t 3W_u \,\mathrm{d}u \,\middle|\, \mathcal{F}_s\right].$$

Conclude that $(W_t^3 - \int_0^t W_u \, \mathrm{d}u)_{t \geqslant 0}$ is a martingale.

Hint: To prove the second equality, you may use that $\sup_{0 \leqslant r \leqslant t} |W_r|$ *is integrable.*

Can you guess a similar result for

$$E[W_t^n - W_s^n \mid \mathcal{F}_s], \quad n \in \mathbb{N}?$$