Mathematical Foundations for Finance

Exercise Sheet 11

Please hand in your solutions by 12:00 on Wednesday, December 7 via the course homepage.

Exercise 11.1 *(Applying Itô's formula)* Let W be a Brownian motion with respect to P and $\mathbb{F}$. Using Itô's formula, write the following as stochastic integrals.

(a) W_t^2,

(b) $t^2 W_t$,

(c) $\sin(2t - W_t)$,

(d) $\exp(at + bW_t)$, where $a, b \in \mathbb{R}$ are constants.

Using part (a), evaluate $\int_0^t W_s \, \mathrm{d}W_s$.

Solution 11.1

(a) Let $f(x) = x^2$. Then f is C^2, and $f'(x) = 2x$ and $f''(x) = 2$. Itô's formula then gives

$$W_t^2 = \int_0^t 2W_s \, \mathrm{d}W_s + \frac{1}{2} \int_0^t 2 \, \mathrm{d}s = 2 \int_0^t W_s \, \mathrm{d}W_s + t.$$

(b) Let $f(t, x) = t^2 x$. Then f is C^2, and $\frac{\partial f}{\partial t}(t, x) = 2tx$, $\frac{\partial f}{\partial x}(t, x) = t^2$ and $\frac{\partial^2 f}{\partial x^2}(t, x) = 0$. Itô's formula then gives

$$t^2 W_t = 2 \int_0^t s W_s \, \mathrm{d}s + \int_0^t s^2 \, \mathrm{d}W_s.$$

(c) Let $f(t, x) = \sin(2t - x)$. Then f is C^2, and we have $\frac{\partial f}{\partial t}(t, x) = 2\cos(2t - x)$, $\frac{\partial f}{\partial x}(t, x) = -\cos(2t - x)$ and $\frac{\partial^2 f}{\partial x^2}(t, x) = -\sin(2t - x)$. We then apply Itô's formula to get

$$\begin{aligned}
\sin(2t - W_t) &= \int_0^t 2\cos(2s - W_s) \, \mathrm{d}s - \int_0^t \cos(2s - W_s) \, \mathrm{d}W_s \\
&\quad - \frac{1}{2} \int_0^t \sin(2s - W_s) \, \mathrm{d}s \\
&= \int_0^t \Big(2\cos(2s - W_s) - \frac{1}{2} \sin(2s - W_s) \Big) \, \mathrm{d}s \\
&\quad - \int_0^t \cos(2s - W_s) \, \mathrm{d}W_s.
\end{aligned}$$

(d) Let $f(t,x) = \exp(at+bx)$. Then f is C^2, and $\frac{\partial f}{\partial t}(t,x) = a\exp(at+bx)$, $\frac{\partial f}{\partial x}(t,x) = b\exp(at+bx)$ and $\frac{\partial^2 f}{\partial x^2}(t,x) = b^2\exp(at+bx)$. Itô's formula then gives

$$
\begin{aligned}
\exp(at+bW_t) &= 1 + \int_0^t a\exp(as+bW_s)\,\mathrm{d}s + \int_0^t b\exp(as+bW_s)\,\mathrm{d}W_s \\
&\quad + \frac{1}{2}\int_0^t b^2\exp(as+bW_s)\,\mathrm{d}s \\
&= 1 + \int_0^t \left(a+\frac{b^2}{2}\right)\exp(as+bW_s)\,\mathrm{d}s \\
&\quad + \int_0^t b\exp(as+bW_s)\,\mathrm{d}W_s,
\end{aligned}
$$

as required.

Finally, rearranging the equality in part (a) gives

$$\int_0^t W_s\,\mathrm{d}W_s = \frac{1}{2}W_t^2 - \frac{1}{2}t.$$

Exercise 11.2 *(Local martingales and Itô's formula)* Let W be a Brownian motion with respect to P and $\mathbb{F}$.

(a) Let $f \in C(\mathbb{R};\mathbb{R})$. Show that the stochastic integral process $(\int_0^t f(W_s)\,\mathrm{d}W_s)_{t\geqslant 0}$ is a continuous local martingale.

(b) Let $f \in C^2(\mathbb{R};\mathbb{R})$. Show that $(f(W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if $\int_0^t f''(W_s)\,\mathrm{d}s = 0$ for all $t \geqslant 0$.

Hint: You may use the fact that a continuous local martingale null at zero is a process of finite variation if and only if it is identically 0.

(c) Let $f \in C^2(\mathbb{R};\mathbb{R})$. Using (b), show that $(f(W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if $f(x) = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$.

Solution 11.2

(a) First note that $(f(W_s))_{s\geqslant 0}$ is adapted (since W is adapted and f is continuous) with continuous paths (since W has continuous paths and f is continuous). In particular, $(f(W_s))_{s\geqslant 0}$ is predictable and locally bounded, and thus belongs to $L^2_{\mathrm{loc}}(W)$. Since W is a (local) martingale null at zero, the stochastic integral process $(\int_0^t f(W_s)\,\mathrm{d}W_s)_{s\geqslant 0}$ is thus a well-defined continuous local martingale.

(b) By Itô's formula,

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\,\mathrm{d}W_s + \frac{1}{2}\int_0^t f''(W_s)\,\mathrm{d}s.$$

By part (a), we know that $(\int_0^t f'(W_s)\ \mathrm{d}W_s)_{t\geqslant 0}$ is a continuous local martingale, and thus $(f(W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if $(\int_0^t f''(W_s)\ \mathrm{d}s)_{t\geqslant 0}$ is also a continuous local martingale. But $(\int_0^t f''(W_s)\ \mathrm{d}s)_{t\geqslant 0}$ is a process of finite variation (indeed, for each $t \geqslant 0$, we have the equality $\int_0^t f''(W_s)\ \mathrm{d}s = \int_0^t f''(W_s)^+\ \mathrm{d}s - \int_0^t f''(W_s)^-\ \mathrm{d}s$, so that $(\int_0^t f''(W_s)\ \mathrm{d}s)_{t\geqslant 0}$ is the difference of two increasing processes), null at zero, and is thus a continuous local martingale if and only if it is identically zero. That is, $(f(W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if $\int_0^t f''(W_s)\ \mathrm{d}s = 0$ for all $t \geqslant 0$, as required.

(c) If $\int_0^t f''(W_s)\ \mathrm{d}s = 0$ P-a.s. for all $t \geqslant 0$, then $f''(W_t) = 0$ P-a.s. for all $t \geqslant 0$. So for all $a \leqslant b \in \mathbb{R}$,

$$0 = E\left[f''(W_t)\mathbb{1}_{W_t\in[a,b]}\right] = \int_a^b f''(x)\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\ \mathrm{d}x.$$

Thus,

$$0 = \lim_{b\downarrow a}\frac{1}{b-a}\int_a^b f''(x)\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\ \mathrm{d}x = f''(a)\frac{1}{\sqrt{2\pi t}}e^{-\frac{a^2}{2t}},$$

and hence $f''(a) = 0$. Since $a \in \mathbb{R}$ was arbitrary, this implies that $f'' \equiv 0$, and so $f' = \alpha$ for some $\alpha \in \mathbb{R}$, and hence $f = \alpha x + \beta$ for some $\beta \in \mathbb{R}$.

Conversely, suppose that $f(x) = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$. Then $f'' = 0$, and thus

$$\int_0^t f''(W_s)\ \mathrm{d}s = 0, \qquad \forall t \geqslant 0.$$

This completes the proof.

Exercise 11.3 *(From local martingale to martingale)* Let M be a local martingale, and assume that $\sup_{0\leqslant t\leqslant T}|M_t| \in L^1$ for some fixed $T > 0$. Prove that M is a martingale on $[0,T]$.

Note. This implies that if $\sup_{0\leqslant t\leqslant T}|M_t| \in L^1$ *for all* $T > 0$*, then* M *is a martingale (on* $[0,\infty)$*).*

Solution 11.3 First, M is adapted since it is a continuous local martingale, and it is integrable since for each $t \in [0,T]$, $|M_t| \leqslant \sup_{0\leqslant t\leqslant T}|M_t| \in L^1$, so that

$$E[|M_t|] \leqslant E\left[\sup_{0\leqslant t\leqslant T}|M_t|\right] < \infty.$$

Finally, fix $0 \leqslant s < t \leqslant T$. We need to show that

$$E[M_t \mid \mathcal{F}_s] = M_s.$$

To this end, let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence of M. Then we have

$$E[M_{t\wedge\tau_n} \mid \mathcal{F}_s] = M_{s\wedge\tau_n}, \qquad \forall n \in \mathbb{N}. \tag{1}$$

Since $\tau_n \uparrow \infty$ as $n \to \infty$, then

$$M_{s\wedge\tau_n} \to M_s \text{ as } n \to \infty \qquad \text{and} \qquad M_{t\wedge\tau_n} \to M_t \text{ as } n \to \infty.$$

Moreover, $|M_{t\wedge\tau_n}| \leqslant \sup_{0\leqslant t\leqslant T} |M_t| \in L^1$ for all n, so we may apply the (conditional) dominated convergence theorem to get

$$E[M_{t\wedge\tau_n} \mid \mathcal{F}_s] \to E[M_t \mid \mathcal{F}_s] \text{ as } n \to \infty.$$

Thus, taking limits as $n \to \infty$ in (1) yields $E[M_t \mid \mathcal{F}_s] = M_s$. Since $0 \leqslant s < t \leqslant T$ were arbitrary, it follows that M is a martingale on $[0, T]$, as required.

Exercise 11.4 *(Martingales and Itô's formula)* Let W be a Brownian motion with respect to P and $\mathbb{F}$. Using Itô's formula, establish which of the following processes are martingales:

(a) $(\sin W_t - \cos W_t)_{t\geqslant 0}$,

(b) $(\exp(\frac{1}{2}a^2 t)\cos(aW_t - b))_{t\geqslant 0}$, where $a, b \in \mathbb{R}$ are constants,

(c) $(W_t^3 - 3tW_t)_{t\geqslant 0}$.

Hint: You may use the fact that for any $T > 0$, the random variables $\max_{0\leqslant t\leqslant T} W_t$ and $-\min_{0\leqslant t\leqslant T} W_t$ have the same distribution as $|W_T|$.

Solution 11.4 By the same reasoning as in Exercise 11.2, we can show using Itô's formula that for $f \in C^2([0,\infty) \times \mathbb{R}; \mathbb{R})$, the process $(f(t, W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if

$$\int_0^t \frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s)\,\mathrm{d}s = 0, \qquad \forall t \geqslant 0.$$

(a) Let $f(x) = \sin x - \cos x$. Then f is C^2, and $f(W_t) = \sin W_t - \cos W_t$. Since $f''(x) = -\sin x + \cos x$, then $f'' \not\equiv 0$, and thus Exercise 11.2(c) gives that $(\sin W_t - \cos W_t)_{t\geqslant 0}$ is not a local martingale. Hence, it is also not a martingale.

(b) Let $f(t, x) = \exp(\frac{1}{2}a^2 t)\cos(ax - b)$. Then f is C^2, and also we have that $\frac{\partial f}{\partial t}(t, x) = \frac{1}{2}a^2 \exp(\frac{1}{2}a^2 t)\cos(ax - b)$ and $\frac{\partial^2 f}{\partial x^2}(t, x) = -a^2 \exp(\frac{1}{2}a^2 t)\cos(ax - b)$. Thus,

$$\begin{aligned}
&\int_0^t \frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s)\,\mathrm{d}s \\
&= \int_0^t \frac{1}{2}a^2 \exp\left(\frac{1}{2}a^2 t\right)\cos(aW_t - b) - \frac{1}{2}a^2 \exp\left(\frac{1}{2}a^2 t\right)\cos(aW_t - b)\,\mathrm{d}s \\
&= 0,
\end{aligned}$$

for all $t \geqslant 0$. Since $\exp(\frac{1}{2}a^2t)\cos(aW_t - b) = f(t, W_t)$, it now follows that $(\exp(\frac{1}{2}a^2t)\cos(aW_t - b))_{t\geqslant 0}$ is a continuous local martingale. Moreover, for $T > 0$,

$$\max_{0\leqslant t\leqslant T}\left|\exp\left(\frac{1}{2}a^2t\right)\cos(aW_t - b)\right| \leqslant \exp\left(\frac{1}{2}a^2T\right) \in L^1.$$

It follows immediately from Exercise 11.3 that $(\exp(\frac{1}{2}a^2t)\cos(aW_t - b))_{t\geqslant 0}$ is a martingale.

(c) Let $f(t, x) = x^3 - 3tx$. Then f is C^2, and $\frac{\partial f}{\partial t}(t, x) = -3x$ and $\frac{\partial^2 f}{\partial x^2}(t, x) = 6x$. We then have that

$$\int_0^t \frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s)\ \mathrm{d}s = \int_0^t 3W_s - 3W_s\ \mathrm{d}s = 0,$$

for all $t \geqslant 0$. Since $W_t^3 - 3tW_t = f(t, W_t)$, it follows that $(W_t^3 - 3tW_t)_{t\geqslant 0}$ is a continuous local martingale. Moreover, for $T > 0$,

$$\max_{0\leqslant t\leqslant T}|W_t^3 - 3tW_t| \leqslant \left(\max_{0\leqslant t\leqslant T}|W_t|\right)^3 + 3T\max_{0\leqslant t\leqslant T}|W_t|.$$

Note that

$$\max_{0\leqslant t\leqslant T}|W_t| \leqslant \max_{0\leqslant t\leqslant T}W_t - \min_{0\leqslant t\leqslant T}W_t.$$

Using the hint, and the fact that $|W_T| \in L^3$, we get that $\max_{0\leqslant t\leqslant T}|W_t| \in L^3$ (and hence also $\max_{0\leqslant t\leqslant T}|W_t| \in L^1$), so that

$$\max_{0\leqslant t\leqslant T}|W_t^3 - 3tW_t| \in L^1.$$

Since $T > 0$ was arbitrary, Exercise 11.3 then gives that $(W_t^3 - 3tW_t)_{t\geqslant 0}$ is a martingale.