Mathematical Foundations for Finance

Exercise Sheet 12

Please hand in your solutions by 12:00 on Wednesday, December 14 via the course homepage.

Exercise 12.1 *(Stochastic product rule)* Let X and Y be two continuous semimartingales. Show that for all $t \geqslant 0$,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t Y_s \, \mathrm{d}X_s + [X, Y]_t.$$

How does this compare to the "classical" product rule from calculus?

Exercise 12.2 *(Stochastic exponential)* Let $X = (X_t)_{t \geqslant 0}$ be a continuous semimartingale. Define the process $\mathcal{E}(X)$ by

$$\mathcal{E}(X)_t := \exp\left(X_t - \frac{1}{2}[X]_t\right).$$

(a) Show that $\mathcal{E}(X)$ is a solution to the SDE

$$Z_t = e^{X_0} + \int_0^t Z_s \, \mathrm{d}X_s, \qquad \forall t \geqslant 0. \tag{1}$$

(b) Prove that $\mathcal{E}(X)$ is the unique solution of (1).

Hint: For a solution Z of (1), consider the process $\frac{Z}{\mathcal{E}(X)}$.

(c) Let $Y = (Y_t)_{t \geqslant 0}$ be another continuous semimartingale. Prove *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Exercise 12.3 *(Itô process)* Let W be a Brownian motion with respect to P and $\mathbb{F}$. An *Itô process* is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}W_s, \qquad t \geqslant 0,$$

where μ and σ are predictable processes (satisfying appropriate integrability conditions). Show that for any C^2 function f, the process $f(X)$ is again an Itô process, and give its decomposition.

Exercise 12.4 *(Distribution of stochastic integral)* Let $g : [0, \infty) \to \mathbb{R}$ be a continuous function. Show that for each $t \geqslant 0$, the random variable

$$X_t := \int_0^t g(s) \, \mathrm{d}W_s$$

is normally distributed, and find its mean and variance.

Hint: For each fixed $\eta \in \mathbb{R}$, show that the stochastic process $Z = (Z_t)_{t \geqslant 0}$, given by $Z_t := e^{-\frac{\eta^2}{2} \int_0^t g^2(s) \, \mathrm{d}s + \eta \int_0^t g(s) \, \mathrm{d}W_s}$, is a local martingale (you may use without proof that Z is actually a true martingale), and then compute $E[e^{\eta \int_0^t g(s) \, \mathrm{d}W_s}]$.