Mathematical Foundations for Finance

Exercise Sheet 12

Please hand in your solutions by 12:00 on Wednesday, December 14 via the course homepage.

Exercise 12.1 *(Stochastic product rule)* Let X and Y be two continuous semimartingales. Show that for all $t \geqslant 0$,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t Y_s \, \mathrm{d}X_s + [X, Y]_t.$$

How does this compare to the "classical" product rule from calculus?

Solution 12.1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = xy$. Then $f \in C^2(\mathbb{R}^2; \mathbb{R})$, and $\frac{\partial f}{\partial x}(x, y) = y$, $\frac{\partial f}{\partial y}(x, y) = x$, $\frac{\partial^2 f}{\partial x^2}(x, y) = 0$, $\frac{\partial^2 f}{\partial x \partial y}(x, y) = 1$, and $\frac{\partial^2 f}{\partial y^2}(x, y) = 0$. Noting that $f(X_t, Y_t) = X_t Y_t$, Itô's formula then gives

$$\begin{aligned} X_t Y_t - X_0 Y_0 &= \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t Y_s \, \mathrm{d}X_s + \int_0^t \, \mathrm{d}[X, Y]_s \\ &= \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t Y_s \, \mathrm{d}X_s + [X, Y]_t, \end{aligned}$$

as required.

The above equality has the extra "$[X, Y]_t$" term, which vanishes in classical calculus.

Exercise 12.2 *(Stochastic exponential)* Let $X = (X_t)_{t \geqslant 0}$ be a continuous semimartingale. Define the process $\mathcal{E}(X)$ by

$$\mathcal{E}(X)_t := \exp\left(X_t - \frac{1}{2}[X]_t\right).$$

(a) Show that $\mathcal{E}(X)$ is a solution to the SDE

$$Z_t = e^{X_0} + \int_0^t Z_s \, \mathrm{d}X_s, \qquad \forall t \geqslant 0. \tag{1}$$

(b) Prove that $\mathcal{E}(X)$ is the unique solution of (1).

Hint: For a solution Z of (1), consider the process $\frac{Z}{\mathcal{E}(X)}$.

(c) Let $Y = (Y_t)_{t \geqslant 0}$ be another continuous semimartingale. Prove *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Solution 12.2

(a) Let $f(x, y) = \exp(x - \frac{1}{2}y)$. Then f is C^2, and we have $\frac{\partial f}{\partial x}(x, y) = f(x, y)$, $\frac{\partial f}{\partial y}(x, y) = -\frac{1}{2}f(x, y)$, and $\frac{\partial f^2}{\partial x^2}(x, y) = f(x, y)$. Noting that $\mathcal{E}(X)_t = f(X_t, [X]_t)$ (and $[[X]]_t = [X, [X]]_t = [[X], X]_t = 0$ because X is continuous), we apply Itô's formula to get

$$\begin{aligned}\mathcal{E}(X)_t &= \mathcal{E}(X)_0 + \int_0^t \mathcal{E}(X)_s \,\mathrm{d}X_s + \int_0^t \left(-\frac{1}{2}\mathcal{E}(X)_s\right) \mathrm{d}[X]_s + \frac{1}{2}\int_0^t \mathcal{E}(X)_s \,\mathrm{d}[X]_s \\ &= e^{X_0} + \int_0^t \mathcal{E}(X)_s \,\mathrm{d}X_s.\end{aligned}$$

It follows immediately that $\mathcal{E}(X)$ is a solution of (1).

(b) Suppose that Z is another solution to (1), and define $Y_t := \frac{Z_t}{\mathcal{E}(X)_t}$. Consider the function $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ defined by $f(x, y) = \frac{x}{y}$. Then f is C^2, and $\frac{\partial f}{\partial x} = \frac{1}{y}$, $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2}$, and $\frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3}$. Noting that $Y_t = f(Z_t, \mathcal{E}(X)_t)$, Itô's formula then gives

$$\begin{aligned}Y_t = Y_0 &+ \int_0^t \frac{1}{\mathcal{E}(X)_s} \,\mathrm{d}Z_s + \int_0^t \left(-\frac{Z_s}{\mathcal{E}(X)_s^2}\right) \mathrm{d}\mathcal{E}(X)_t \\ &- \int_0^t \frac{1}{\mathcal{E}(X)_s^2} \,\mathrm{d}[Z, \mathcal{E}(X)]_s + \frac{1}{2}\int_0^t \frac{2Z_s}{\mathcal{E}(X)_t^3} \,\mathrm{d}[Z]_s.\end{aligned}$$

Since $\mathcal{E}(X)$ and Z are solutions of (1), $\mathrm{d}\mathcal{E}(X)_t = \mathcal{E}(X)_t \,\mathrm{d}X_s$ and $\mathrm{d}Z_t = Z_t \,\mathrm{d}X_s$, and so $\mathrm{d}[Z, \mathcal{E}(X)]_t = Z_t\mathcal{E}(X)_t \,\mathrm{d}[X]_s$ and $\mathrm{d}[Z]_t = Z_t^2 \,\mathrm{d}[X]_s$. The above can now be rewritten as

$$\begin{aligned}Y_t = Y_0 &+ \int_0^t \frac{Z_s}{\mathcal{E}(X)_s} \,\mathrm{d}X_s + \int_0^t \left(-\frac{Z_s}{\mathcal{E}(X)_s^2}\mathcal{E}(X)_s\right) \mathrm{d}X_s \\ &- \int_0^t \frac{1}{\mathcal{E}(X)_s^2} Z_s\mathcal{E}(X)_s \,\mathrm{d}[X]_s + \frac{1}{2}\int_0^t \frac{2Z_s}{\mathcal{E}(X)_s^3}\mathcal{E}(X)_s^2 \,\mathrm{d}[X]_s \\ = Y_0&.\end{aligned}$$

Since $\mathcal{E}(X)_0 = e^{X_0} = Z_0$, then $Y_0 = 1$, and hence we have

$$Y \equiv 1.$$

It follows immediately that $Z = \mathcal{E}(X)$, and thus $\mathcal{E}(X)$ is the unique solution of (1).

(c) The stochastic product rule gives

$$\mathcal{E}(X)_t\mathcal{E}(Y)_t = e^{X_0}e^{Y_0} + \int_0^t \mathcal{E}(X)_s \,\mathrm{d}\mathcal{E}(Y)_s + \int_0^t \mathcal{E}(Y)_s \,\mathrm{d}\mathcal{E}(X)_s + [\mathcal{E}(X), \mathcal{E}(Y)]_t.$$

Using part (a), we have $\mathrm{d}\mathcal{E}(X)_t = \mathcal{E}(X)_t \,\mathrm{d}X_t$ and $\mathrm{d}\mathcal{E}(Y)_t = \mathcal{E}(Y)_t \,\mathrm{d}Y_t$, and so $\mathrm{d}[\mathcal{E}(X), \mathcal{E}(Y)]_t = \mathcal{E}(X)_t\mathcal{E}(Y)_t \,\mathrm{d}[X, Y]_t$. Thus, we can rewrite the above as

$$\begin{aligned}\mathcal{E}(X)_t\mathcal{E}(Y)_t &= e^{X_0}e^{Y_0} + \int_0^t \mathcal{E}(X)_s\mathcal{E}(Y)_s \,\mathrm{d}Y_s + \int_0^t \mathcal{E}(Y)_s\mathcal{E}(X)_s \,\mathrm{d}X_s \\ &\quad + \int_0^t \mathcal{E}(X)_s\mathcal{E}(Y)_s \,\mathrm{d}[X, Y]_s \\ &= e^{X_0}e^{Y_0} + \int_0^t \mathcal{E}(X)_s\mathcal{E}(Y)_s \,\mathrm{d}(X + Y + [X, Y])_s.\end{aligned}$$

We have thus shown that $\mathcal{E}(X)\mathcal{E}(Y)$ is a solution to the SDE

$$Z_t = e^{(X+Y+[X,Y])_0} + \int_0^t Z_s \,\mathrm{d}(X + Y + [X, Y])_s, \qquad \forall t \geqslant 0.$$

By part (b), it follows immediately that

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]),$$

as required.

Exercise 12.3 *(Itô process)* Let W be a Brownian motion with respect to P and $\mathbb{F}$. An *Itô process* is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}W_s, \qquad t \geqslant 0,$$

where μ and σ are predictable processes (satisfying appropriate integrability conditions). Show that for any C^2 function f, the process $f(X)$ is again an Itô process, and give its decomposition.

Solution 12.3 By Itô's formula, we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s) \,\mathrm{d}[X]_s.$$

We know that $\mathrm{d}X_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t$, and hence $\mathrm{d}[X]_t = \sigma_t^2 \,\mathrm{d}t$. The above equality can thus be rewritten as

$$\begin{aligned}f(X_t) &= f(X_0) + \int_0^t f'(X_s)\mu_s \,\mathrm{d}s + \int_0^t f'(X_s)\sigma_s \,\mathrm{d}W_s + \frac{1}{2}\int_0^t f''(X_s)\sigma_s^2 \,\mathrm{d}s \\ &= f(X_0) + \int_0^t \Big(\mu_s f'(X_s) + \frac{1}{2}\sigma_s^2 f''(X_s)\Big) \,\mathrm{d}s + \int_0^t \sigma_s f'(X_s) \,\mathrm{d}W_s.\end{aligned}$$

Thus, $f(X)$ is an Itô process with the decomposition

$$\widetilde{\mu}_s = \mu_s f'(X_s) + \frac{1}{2}\sigma_s^2 f''(X_s) \qquad \text{and} \qquad \widetilde{\sigma}_s = \sigma_s f'(X_s).$$

Exercise 12.4 *(Distribution of stochastic integral)* Let $g : [0, \infty) \to \mathbb{R}$ be a continuous function. Show that for each $t \geqslant 0$, the random variable

$$X_t := \int_0^t g(s) \, \mathrm{d}W_s$$

is normally distributed, and find its mean and variance.

Hint: For each fixed $\eta \in \mathbb{R}$, show that the stochastic process $Z = (Z_t)_{t \geqslant 0}$, given by $Z_t := e^{-\frac{\eta^2}{2}\int_0^t g^2(s)\,\mathrm{d}s + \eta \int_0^t g(s)\,\mathrm{d}W_s}$, is a local martingale (you may use without proof that Z is actually a true martingale), and then compute $E[e^{\eta \int_0^t g(s)\,\mathrm{d}W_s}]$.

Solution 12.4 Following the hint, we first show that Z is a martingale. Letting $Y_t := \int_0^t g^2(s) \, \mathrm{d}s$ (which is well-defined because g is bounded on $[0, t]$, since g is continuous), we can write

$$Z_t = e^{-\frac{\eta^2}{2} Y_t + \eta X_t}.$$

Let $f(x, y) = e^{-\frac{\eta^2}{2} y + \eta x}$. Then $f \in C^2(\mathbb{R}^2; \mathbb{R})$, and $\frac{\partial f}{\partial x} = \eta f$, $\frac{\partial f}{\partial y} = -\frac{\eta^2}{2} f$, $\frac{\partial^2 f}{\partial x^2} = \eta^2 f$, $\frac{\partial^2 f}{\partial x \partial y} = -\frac{\eta^3}{2} f$, and $\frac{\partial^2 f}{\partial y^2} = \frac{\eta^4}{4} f$. Noting that $Z_t = f(X_t, Y_t)$, we apply Itô's formula to get

$$\begin{aligned} Z_t = Z_0 &+ \int_0^t \eta Z_s \, \mathrm{d}X_s + \int_0^t \left(-\frac{\eta^2}{2} Z_s\right) \mathrm{d}Y_s \\ &+ \frac{1}{2}\int_0^t \eta^2 Z_s \, \mathrm{d}[X]_s + \int_0^t \left(-\frac{\eta^3}{2} Z_s\right) \mathrm{d}[X, Y]_s + \frac{1}{2}\int_0^t \frac{\eta^4}{4} Z_s \, \mathrm{d}[Y]_s. \end{aligned}$$

The equalities $X_t = \int_0^t g(s) \, \mathrm{d}W_s$ and $Y_t = \int_0^t g^2(s) \, \mathrm{d}s$ can be equivalently written as $\mathrm{d}X_t = g(t) \, \mathrm{d}W_t$ and $\mathrm{d}Y_t = g^2(t) \, \mathrm{d}t$. We then also have $\mathrm{d}[X]_t = g^2(t) \, \mathrm{d}t$, $\mathrm{d}[X, Y]_t = 0$, and $\mathrm{d}[Y]_t = 0$. We can then rewrite the above as

$$\begin{aligned} Z_t &= Z_0 + \int_0^t \eta Z_s g(s) \, \mathrm{d}W_s + \int_0^t \left(-\frac{\eta^2}{2} Z_s g^2(s)\right) \mathrm{d}s + \frac{1}{2}\int_0^t \eta^2 Z_s g^2(s) \, \mathrm{d}s \\ &= Z_0 + \int_0^t \eta Z_s g(s) \, \mathrm{d}W_s. \end{aligned}$$

It follows that Z is a local martingale, and in fact a true martingale, by the hint. In particular, $E[Z_t] = E[Z_0]$. Noting that $Z_0 = 1$, we get

$$1 = E[e^{-\frac{\eta^2}{2} Y_t + \eta X_t}] = e^{-\frac{\eta^2}{2} Y_t} E[e^{\eta X_t}],$$

and thus

$$E[e^{\eta X_t}] = e^{\frac{\eta^2}{2} Y_t}.$$

Thus, the moment generating function of the random variable X_t is given by $e^{\frac{\eta^2}{2} Y_t}$. Since the moment generating function determines the distribution, it follows that X_t is normally distributed with mean zero and variance $Y_t = \int_0^t g^2(s)\,\mathrm{d}s$. This completes the proof.