Mathematical Foundations for Finance

Exercise Sheet 13

Please hand in your solutions by 12:00 on Wednesday, December 21 via the course homepage.

In the problems below, W denotes a Brownian motion with respect to P and $\mathbb{F}$.

Exercise 13.1 *(Novikov condition)* Let $\eta \in \mathbb{R}$ be fixed and $g : \mathbb{R} \to \mathbb{R}$ a continuous function. Recall that in Exercise 12.4, we showed via Itô's formula that the process

$$e^{\eta \int_0^t g(s)\,\mathrm{d}W_s - \frac{\eta^2}{2}\int_0^t g^2(s)\,\mathrm{d}s}, \qquad t \geqslant 0,$$

is a continuous local martingale. Prove that it is in fact a true martingale.

Exercise 13.2 *(Itô's representation theorem)* Fix $T \in [0, \infty)$. For each of the following random variables, determine its decomposition as given in Itô's representation theorem.

(a) W_T,

(b) W_T^4,

(c) $\cos(W_T)$.

Hint: For part (b), compute the martingale M on $[0,T]$ with $M_T = W_T^4$, and for part (c), argue first that $(e^{t/2}\cos(W_t))_{t\geqslant 0}$ is a martingale.

Exercise 13.3 The purpose of this exercise is to establish non-uniqueness of the integrand ψ in the theorem of Dudley (Theorem 6.3.3 in the lecture notes). To this end, find a predictable and bounded process ϕ with $0 < \int_0^\infty \phi_t^2 \,\mathrm{d}t < \infty$ P-a.s., and such that

$$\int_0^\infty \phi_t \,\mathrm{d}W_t = 0.$$

Exercise 13.4 *(Computation for Black–Scholes)* The purpose of this exercise is to perform a computation that will be needed in Chapter 7 of the lecture notes.

Let a, b, c, d be fixed constants with $a, d > 0$, and let Z be a standard normal random variable. Compute

$$E[(ae^{bZ+c} - d)^+],$$

where $x^+ = \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$.