Mathematical Foundations for Finance

Exercise Sheet 13

Please hand in your solutions by 12:00 on Wednesday, December 21 via the course homepage.

In the problems below, W denotes a Brownian motion with respect to P and $\mathbb{F}$.

Exercise 13.1 *(Novikov condition)* Let $\eta \in \mathbb{R}$ be fixed and $g : \mathbb{R} \to \mathbb{R}$ a continuous function. Recall that in Exercise 12.4, we showed via Itô's formula that the process

$$e^{\eta \int_0^t g(s)\, \mathrm{d}W_s - \frac{\eta^2}{2} \int_0^t g^2(s)\, \mathrm{d}s}, \qquad t \geqslant 0,$$

is a continuous local martingale. Prove that it is in fact a true martingale.

Solution 13.1 For ease of notation, let $L_t := \int_0^t \eta g(s)\, \mathrm{d}W_s$ for all $t \geqslant 0$. It is immediate that L is a continuous local martingale null at zero. Also, note that

$$\langle L \rangle_t = \int_0^t \eta^2 g^2(s)\ \mathrm{d}s.$$

Thus,

$$\mathcal{E}(L)_t := \exp(L_t - \tfrac{1}{2}\langle L \rangle_t) = e^{\eta \int_0^t g(s)\, \mathrm{d}W_s - \frac{\eta^2}{2} \int_0^t g^2(s)\, \mathrm{d}s}.$$

Moreover, for each $T > 0$, we have

$$E[e^{\frac{1}{2}\langle L \rangle_T}] = E[e^{\frac{1}{2}\int_0^T \eta^2 g^2(s)\, \mathrm{d}s}] = e^{\frac{1}{2}\int_0^T \eta^2 g^2(s)\, \mathrm{d}s} < \infty,$$

since $s \mapsto \eta g^2(s)$ is continuous on $[0, T]$ (and hence bounded). It now follows from the Novikov criterion that the process given in the problem is a true martingale on $[0, T]$. Since this is true for all $T > 0$, the process is thus a true martingale on $[0, \infty)$, as required.

Exercise 13.2 *(Itô's representation theorem)* Fix $T \in [0, \infty)$. For each of the following random variables, determine its decomposition as given in Itô's representation theorem.

(a) W_T,

(b) W_T^4,

(c) $\cos(W_T)$.

Hint: For part (b), compute the martingale M on $[0,T]$ with $M_T = W_T^4$, and for part (c), argue first that $(e^{t/2}\cos(W_t))_{t\geqslant 0}$ is a martingale.

Solution 13.2

(a) We see that the process $\psi = (\psi_s)_{s\geqslant 0}$, given by

$$\psi_s = \begin{cases} 1 & \text{if } s \leqslant T, \\ 0 & \text{if } s > T, \end{cases}$$

satisfies

$$W_T = \int_0^\infty \psi_s \, \mathrm{d}W_s.$$

Since $(W_t)_{t\in[0,T]}$ is a martingale, then $(\int_0^t \psi_s \, \mathrm{d}W_s)_{t\in[0,\infty]}$ is a martingale, as required.

(b) Note that there is a unique martingale $M = (M_t)_{t\in[0,T]}$ with $M_T = W_T^4$, namely, $M_t = E[W_T^4 \mid \mathcal{F}_t]$ for all $t \in [0,T]$. We want to have $M_t = \int_0^t \psi_s \, \mathrm{d}W_s$ for all $t \in [0,T]$. We compute

$$\begin{aligned} M_t &= E[W_T^4 \mid \mathcal{F}_t] = E[(W_T - W_t + W_t)^4 \mid \mathcal{F}_t] \\ &= W_t^4 + 4W_t^3 E[W_T - W_t \mid \mathcal{F}_t] + 6W_t^2 E[(W_T - W_t)^2 \mid \mathcal{F}_t] \\ &\quad + 4W_t E[(W_T - W_t)^3 \mid \mathcal{F}_t] + E[(W_T - W_t)^4 \mid \mathcal{F}_t]. \end{aligned}$$

Since $W_T - W_t$ is independent of $\mathcal{F}_t$ and has distribution $N(0, T-t)$, then

$$\begin{aligned} E[W_T - W_t \mid \mathcal{F}_t] &= E[W_T - W_t] = 0, \\ E[(W_T - W_t)^2 \mid \mathcal{F}_t] &= E[(W_T - W_t)^2] = T - t, \\ E[(W_T - W_t)^3 \mid \mathcal{F}_t] &= E[(W_T - W_t)^3] = 0, \\ E[(W_T - W_t)^4 \mid \mathcal{F}_t] &= E[(W_T - W_t)^4] = 3(T-t)^2. \end{aligned}$$

Thus, we have

$$M_t = W_t^4 + 6(T-t)W_t^2 + 3(T-t)^2.$$

Now, let $f(t,x) = x^4 + 6(T-t)x^2 + 3(T-t)^2$. Then $f \in C^2(\mathbb{R}^2;\mathbb{R})$, and $\frac{\partial f}{\partial t} = -6x^2 - 6(T-t)$, $\frac{\partial f}{\partial x} = 4x^3 + 12(T-t)x$, and $\frac{\partial^2 f}{\partial x^2} = 12x^2 + 12(T-t)$. By Itô's formula, we have

$$\begin{aligned} M_t &= \int_0^t -6W_s^2 - 6(T-s) \, \mathrm{d}s + \int_0^t 4W_s^3 + 12(T-s)W_t \, \mathrm{d}W_s \\ &\quad + \frac{1}{2}\int_0^t 12W_s^2 + 12(T-s) \, \mathrm{d}s \\ &= \int_0^t 4W_s^3 + 12(T-s)W_t \, \mathrm{d}W_s. \end{aligned}$$

Thus, take

$$\psi_s = \begin{cases} 4W_s^3 + 12(T-s)W_t & \text{if } s \leqslant T, \\ 0 & \text{if } s > T. \end{cases}$$

Then $\int_0^t \psi_s \, \mathrm{d}W_s = M_t$ for all $t \in [0,T]$, and $\int_0^t \psi_s \, \mathrm{d}s = M_T$ for all $t \in [T,\infty]$. Since $(M_t)_{t\in[0,T]}$ is a martingale, then so is $(\int_0^t \psi_s \, \mathrm{d}W_s)_{t\in[0,\infty]}$. Moreover,

$$\int_0^\infty \psi_s \, \mathrm{d}W_s = M_T = W_T^4,$$

as required.

(c) Consider $e^{t/2}\cos(W_t)$, and let $f(t,x) = e^{t/2}\cos(x)$. Then $f \in C^2(\mathbb{R}^2;\mathbb{R})$, and $\frac{\partial f}{\partial t} = \frac{1}{2}e^{t/2}\cos(x)$, $\frac{\partial f}{\partial x} = -e^{t/2}\sin(x)$, $\frac{\partial f^2}{\partial x^2} = -e^{t/2}\cos(x)$. Noting that $e^{t/2}\cos(W_t) = f(t,W_t)$, we apply Itô's formula to get

$$\begin{aligned} e^{t/2}\cos(W_t) &= 1 + \int_0^t \frac{1}{2}e^{s/2}\cos(W_s) \, \mathrm{d}s - \int_0^t e^{s/2}\sin(W_s) \, \mathrm{d}W_s \\ &\quad - \frac{1}{2}\int_0^t e^{s/2}\cos(W_s) \, \mathrm{d}s \\ &= 1 - \int_0^t e^{s/2}\sin(W_s) \, \mathrm{d}W_s. \end{aligned}$$

Thus,

$$\cos(W_t) = e^{-t/2} + \int_0^t -e^{(t-s)/2}\sin(W_s) \, \mathrm{d}W_s.$$

Now take $\psi = (\psi_s)_{s\geqslant 0}$ to be

$$\psi_s = \begin{cases} -e^{(t-s)/2}\sin(W_s) & \text{if } s \leqslant T, \\ 0 & \text{if } s > T. \end{cases}$$

Then

$$\cos(W_T) = e^{-T/2} + \int_0^\infty \psi_s \, \mathrm{d}W_s.$$

Since $E[\int_0^r \psi_s^2 \, \mathrm{d}s] < \infty$ for all $r \in [0,\infty]$, then $(\int_0^r \psi_s \, \mathrm{d}W_s)_{r\in[0,\infty]}$ is a martingale, as required.

Exercise 13.3 The purpose of this exercise is to establish non-uniqueness of the integrand ψ in the theorem of Dudley (Theorem 6.3.3 in the lecture notes). To this end, find a predictable and bounded process ϕ with $0 < \int_0^\infty \phi_t^2 \, \mathrm{d}t < \infty$ P-a.s., and such that

$$\int_0^\infty \phi_t \, \mathrm{d}W_t = 0.$$

Solution 13.3 Consider the stopping time $\tau = \inf\{t \geqslant 1 : W_t = 0\}$. Note that $\tau < \infty$ P-a.s. (e.g. by the law of the iterated logarithm), and also $\tau \geqslant 1$ by construction. Define

$$\phi_t = \mathbb{1}_{[0,t\wedge\tau]}.$$

Then ϕ is adapted and left-continuous, and hence predictable. Clearly ϕ is also bounded. Moreover,

$$\int_0^\infty \phi_t^2 \,\mathrm{d}t = \int_0^\infty \mathbb{1}_{[0,t\wedge\tau]} \,\mathrm{d}t = \int_0^\tau \,\mathrm{d}t = \tau.$$

Since $1 \leqslant \tau < \infty$ P-a.s., then we also have $0 < \int_0^\infty \phi_t^2 \,\mathrm{d}t < \infty$ P-a.s. Next, we compute

$$\int_0^\infty \phi_t \,\mathrm{d}W_t = \int_0^\tau \mathrm{d}W_t = W_\tau = 0.$$

This completes the problem.

Exercise 13.4 *(Computation for Black–Scholes)* The purpose of this exercise is to perform a computation that will be needed in Chapter 7 of the lecture notes.

Let a, b, c, d be fixed constants with $a, d > 0$, and let Z be a standard normal random variable. Compute

$$E[(ae^{bZ+c} - d)^+],$$

where $x^+ = \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$.

Solution 13.4 Using that Z has density $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, we have

$$E[(ae^{bZ+c} - d)^+] = \int_{-\infty}^\infty (ae^{bz+c} - d)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \,\mathrm{d}z.$$

The integrand is nonzero precisely when $ae^{bz+c} - d > 0$, which is equivalent to $z > \frac{1}{b}(\log \frac{d}{a} - c)$. The above integral is thus

$$\begin{aligned}
\int_{\frac{1}{b}(\log \frac{d}{a} - c)}^\infty (ae^{bz+c} - d) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \,\mathrm{d}z &= a \int_{\frac{1}{b}(\log \frac{d}{a} - c)}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 - 2bz - 2c}{2}} \,\mathrm{d}z \\
&\quad - d \int_{\frac{1}{b}(\log \frac{d}{a} - c)}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \,\mathrm{d}z \\
&= ae^{\frac{b^2 - 2c}{2}} \int_{\frac{1}{b}(\log \frac{d}{a} - c)}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-b)^2}{2}} \,\mathrm{d}z \\
&\quad - d \int_{\frac{1}{b}(\log \frac{d}{a} - c)}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \,\mathrm{d}z \\
&= ae^{\frac{b^2 - 2c}{2}} \int_{\frac{1}{b}(\log \frac{d}{a} - b^2 - c)}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \,\mathrm{d}z \\
&\quad - d \int_{\frac{1}{b}(\log \frac{d}{a} - c)}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \,\mathrm{d}z.
\end{aligned}$$

Letting $\Phi(z) = P[Z \leqslant z]$ denote the (cumulative) distribution function of a standard normal random variable, we thus have

$$E[(ae^{bZ+c} - d)^+] = ae^{\frac{b^2-2c}{2}}\Phi\left(\frac{b^2 + c - \log\frac{d}{a}}{b}\right) - d\Phi\left(\frac{c - \log\frac{d}{a}}{b}\right).$$