Mathematical Foundations for Finance

Exercise Sheet 14

This exercise sheet is not for hand-in.

Throughout Exercises 14.1–14.16, there is exactly one correct answer.

Throughout Exercises 14.1–14.8, let $(\widetilde{S}^0, \widetilde{S}^1)$ *be an undiscounted financial market in discrete time on* $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *with a finite time horizon* $T \in \mathbb{N}$ *and* $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ *generated by* $\widetilde{S}^1$*. Let* $\widetilde{S}^0_k := (1+r)^k$ *for* $k = 0, 1, \dots, T$ *with constants* $r > -1$ *and* $\widetilde{S}^1_0 := s^1_0 > 0$*. The discounted market is denoted by* (S^0, S^1)*.*

Exercise 14.1 Which of the following statements is correct?

(a) If the market (S^0, S^1) is complete, then it is arbitrage-free.

(b) If the market (S^0, S^1) is arbitrage-free and complete, then every contingent claim $K \in L^0_+(\mathcal{F}_T)$ admits a unique replicating strategy.

(c) If (S^0, S^1) is arbitrage-free but not complete, then there exists a contingent claim $H \in L^0_+(\mathcal{F}_T)$ that admits infinitely many arbitrage-free price processes.

Exercise 14.2 Suppose that S^1 satisfies $S^1_0 = 1$ and $\log S^1_1 = Z$, where Z is an exponentially distributed random variable, i.e. the density f_Z of Z is given by $f_Z(z) = \exp(-z)$ for $z \geqslant 0$. Then:

(a) The market (S^0, S^1) is arbitrage-free but not complete.

(b) The market (S^0, S^1) admits an arbitrage.

(c) The market (S^0, S^1) is arbitrage-free and complete.

Exercise 14.3 The market (S^0, S^1) is *not* arbitrage-free if

(a) the set of all equivalent local martingale measures for S^1 is non-empty.

(b) there exists a self-financing, admissible strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with $V_T(\varphi) \geqslant 0$ P-a.s. and $P[V_T(\varphi) > 0] > 0$.

(c) S^1 is a $(P, \mathbb{F})$-martingale.

Exercise 14.4 Let τ be an $\mathbb{F}$-stopping time and $X = (X_k)_{k=0,1,\dots,T}$ a $(P, \mathbb{F})$-martingale. Then:

(a) $X^\tau = (X_k^\tau)_{k=0,1,\dots,T}$ is a $(P,\mathbb{F})$-martingale.

(b) $X^\tau = (X_k^\tau)_{k=0,1,\dots,T}$ is a $(P,\mathbb{F})$-supermartingale, but not a $(P,\mathbb{F})$-martingale.

(c) $X^\tau = (X_k^\tau)_{k=0,1,\dots,T}$ is a $(P,\mathbb{F})$-submartingale, but not a $(P,\mathbb{F})$-martingale.

Exercise 14.5 Let τ_1, τ_2 be two stopping times. Which of the following is a stopping time?

(a) $\lfloor \frac{\tau_1+\tau_2}{2} \rfloor$, where for any real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

(b) $\tau_1 \mathbb{1}_{\{\tau_1 \geqslant \tau_2\}}$.

(c) $\mathbb{1}_{\{\tau_1 > 0\}}$.

Exercise 14.6 Which of the following conditions does not imply that $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free?

(a) There exists a probability measure $Q \approx P$ such that $aS^0 + bS^1$ is a $(Q,\mathbb{F})$-martingale for any $a, b \in \mathbb{R}$.

(b) $\widetilde{S}^0/\widetilde{S}^1$ is a positive martingale.

(c) There exists a positive martingale Z such that ZS^1 is a martingale.

Exercise 14.7 Which of the following statements is not true about the binomial model?

(a) The market is complete if it is arbitrage-free.

(b) Every strategy is admissible.

(c) Every strategy is self-financing.

Exercise 14.8 Let $M = (M_k)_{k\in\mathbb{N}_0}$ be an adapted process. Which of the following does not imply that M is a martingale?

(a) M is a supermartingale such that $E[M_k]$ is an increasing sequence.

(b) M is integrable and for each $k = 1, \dots, T-1$, $E[M_k - M_{k+1} \mid \mathcal{F}_{k-1}] = 0$.

(c) M is a bounded local martingale.

Throughout Exercises 14.9–14.16, W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfies the usual conditions of right-continuity and P-completeness.

Exercise 14.9 Let M be a local $(P, \mathbb{F})$-martingale and A a process of finite variation. Then:

(a) $H \bullet M$ is a martingale if H is bounded and predictable.

(b) $H \bullet A$ has finite variation if H is locally bounded and predictable.

(c) $H \bullet M$ is a martingale if M is a martingale and H is locally bounded and predictable.

Exercise 14.10 Suppose $(\widetilde{S}^0, \widetilde{S}^1)$ is the undiscounted Black–Scholes model with parameters $\sigma > 0$ and $\mu < r$, $\mu, r \in \mathbb{R}$. Then:

(a) There is no equivalent martingale measure for $\widetilde{S}^1/\widetilde{S}^0$.

(b) There is an equivalent martingale measure for $\widetilde{S}^0/\widetilde{S}^1$.

(c) None of the above.

Exercise 14.11 Suppose that $Q \approx P$ with density $\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp(W_t - t/2)$. Which of the following is not true?

(a) $W_t^2 - t$, $t \geqslant 0$, is a Q-martingale.

(b) $\exp(W_t - 3t/2)$, $t \geqslant 0$, is a Q-martingale.

(c) $f(W_t)$, $t \geqslant 0$, is a $(Q, \mathbb{F})$-semimartingale for $f \in C^2(\mathbb{R})$.

Exercise 14.12 Consider the Black–Scholes model. Then:

(a) The price of a European put option increases if the strike is decreased.

(b) The price of a European put option increases if the interest rate is increased.

(c) None of the above.

Exercise 14.13 Let M be a local $(P, \mathbb{F})$-martingale and A a process of finite variation. Then:

(a) $[M, A] = 0$ P-a.s.

(b) $[M, A] = 0$ P-a.s. if and only if both M and A are continuous.

(c) $[M, A] = 0$ P-a.s. if A is continuous.

Exercise 14.14 Let $Z = (Z_t)_{t \in [0,T]}$ for some $T \in (0, \infty)$ be the density process of $Q \approx P$ on $\mathcal{F}_T$ with respect to P. Which of the following is true?

(a) $Z = Z_0\,\mathcal{E}(L)$ for some continuous local $(P, \mathbb{F})$-martingale $L = (L_t)_{t\in[0,T]}$ null at 0.

(b) $Z = Z_0\,\mathcal{E}(L)$ for some local $(P, \mathbb{F})$-martingale $L = (L_t)_{t\in[0,T]}$ null at 0.

(c) $Z = Z_0\,\mathcal{E}(L)$ for some local $(Q, \mathbb{F})$-martingale $L = (L_t)_{t\in[0,T]}$ null at 0.

Exercise 14.15 Which of the following statements about W is not true?

(a) W has infinite 1-variation P-a.s.

(b) W is a $(P, \mathbb{F})$-semimartingale.

(c) W is the unique continuous process with zero mean and normally distributed increments.

Exercise 14.16 Let X be a $(P, \mathbb{F})$-semimartingale and $Q \overset{\text{loc}}{\approx} P$. Then:

(a) $f(X)$ is a $(Q, \mathbb{F})$-semimartingale for any C^∞-function f.

(b) X is a $(Q, \mathbb{F})$-martingale.

(c) $f(X)$ is a $(P, \mathbb{F})$-semimartingale for any measurable function f.