

Mathematical Foundations for Finance

Exercise Sheet 14

This exercise sheet is not for hand-in.

Throughout Exercises 14.1–14.16, there is exactly one correct answer.

Throughout Exercises 14.1–14.8, let $(\tilde{S}^0, \tilde{S}^1)$ be an undiscounted financial market in discrete time on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a finite time horizon $T \in \mathbb{N}$ and $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by \tilde{S}^1 . Let $\tilde{S}_k^0 := (1+r)^k$ for $k = 0, 1, \dots, T$ with constants $r > -1$ and $\tilde{S}_0^1 := s_0^1 > 0$. The discounted market is denoted by (S^0, S^1) .

Exercise 14.1 Which of the following statements is correct?

- (a) If the market (S^0, S^1) is complete, then it is arbitrage-free.
- (b) If the market (S^0, S^1) is arbitrage-free and complete, then every contingent claim $K \in L_+^0(\mathcal{F}_T)$ admits a unique replicating strategy.
- (c) If (S^0, S^1) is arbitrage-free but not complete, then there exists a contingent claim $H \in L_+^0(\mathcal{F}_T)$ that admits infinitely many arbitrage-free price processes.

Solution 14.1 (c).

Exercise 14.2 Suppose that S^1 satisfies $S_0^1 = 1$ and $\log S_1^1 = Z$, where Z is an exponentially distributed random variable, i.e. the density f_Z of Z is given by $f_Z(z) = \exp(-z)$ for $z \geq 0$. Then:

- (a) The market (S^0, S^1) is arbitrage-free but not complete.
- (b) The market (S^0, S^1) admits an arbitrage.
- (c) The market (S^0, S^1) is arbitrage-free and complete.

Solution 14.2 (b).

Exercise 14.3 The market (S^0, S^1) is *not* arbitrage-free if

- (a) the set of all equivalent local martingale measures for S^1 is non-empty.
- (b) there exists a self-financing, admissible strategy $\varphi \equiv (0, \vartheta)$ with $V_T(\varphi) \geq 0$ P -a.s. and $P[V_T(\varphi) > 0] > 0$.

(c) S^1 is a (P, \mathbb{F}) -martingale.

Solution 14.3 (b).

Exercise 14.4 Let τ be an \mathbb{F} -stopping time and $X = (X_k)_{k=0,1,\dots,T}$ a (P, \mathbb{F}) -martingale. Then:

- (a) $X^\tau = (X_k^\tau)_{k=0,1,\dots,T}$ is a (P, \mathbb{F}) -martingale.
- (b) $X^\tau = (X_k^\tau)_{k=0,1,\dots,T}$ is a (P, \mathbb{F}) -supermartingale, but not a (P, \mathbb{F}) -martingale.
- (c) $X^\tau = (X_k^\tau)_{k=0,1,\dots,T}$ is a (P, \mathbb{F}) -submartingale, but not a (P, \mathbb{F}) -martingale.

Solution 14.4 (a).

Exercise 14.5 Let τ_1, τ_2 be two stopping times. Which of the following is a stopping time?

- (a) $\lfloor \frac{\tau_1 + \tau_2}{2} \rfloor$, where for any real number x , $\lfloor x \rfloor$ is the greatest integer less than or equal to x .
- (b) $\tau_1 \mathbb{1}_{\{\tau_1 \geq \tau_2\}}$.
- (c) $\mathbb{1}_{\{\tau_1 > 0\}}$.

Solution 14.5 (c).

Exercise 14.6 Which of the following conditions does not imply that $(\tilde{S}^0, \tilde{S}^1)$ is arbitrage-free?

- (a) There exists a probability measure $Q \approx P$ such that $aS^0 + bS^1$ is a (Q, \mathbb{F}) -martingale for any $a, b \in \mathbb{R}$.
- (b) \tilde{S}^0/\tilde{S}^1 is a positive martingale.
- (c) There exists a positive martingale Z such that ZS^1 is a martingale.

Solution 14.6 (a).

Exercise 14.7 Which of the following statements is not true about the binomial model?

- (a) The market is complete if it is arbitrage-free.
- (b) Every strategy is admissible.

- (c) Every strategy is self-financing.

Solution 14.7 (c).

Exercise 14.8 Let $M = (M_k)_{k \in \mathbb{N}_0}$ be an adapted process. Which of the following does not imply that M is a martingale?

- (a) M is a supermartingale such that $E[M_k]$ is an increasing sequence.
 (b) M is integrable and for each $k = 1, \dots, T - 1$, $E[M_k - M_{k+1} \mid \mathcal{F}_{k-1}] = 0$.
 (c) M is a bounded local martingale.

Solution 14.8 (b).

Throughout Exercises 14.9–14.16, W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions of right-continuity and P -completeness.

Exercise 14.9 Let M be a local (P, \mathbb{F}) -martingale and A a process of finite variation. Then:

- (a) $H \bullet M$ is a martingale if H is bounded and predictable.
 (b) $H \bullet A$ has finite variation if H is locally bounded and predictable.
 (c) $H \bullet M$ is a martingale if M is a martingale and H is locally bounded and predictable.

Solution 14.9 (b).

Exercise 14.10 Suppose $(\tilde{S}^0, \tilde{S}^1)$ is the undiscounted Black–Scholes model with parameters $\sigma > 0$ and $\mu < r$, $\mu, r \in \mathbb{R}$. Then:

- (a) There is no equivalent martingale measure for \tilde{S}^1/\tilde{S}^0 .
 (b) There is an equivalent martingale measure for \tilde{S}^0/\tilde{S}^1 .
 (c) None of the above.

Solution 14.10 (b).

Exercise 14.11 Suppose that $Q \approx P$ with density $\frac{dQ}{dP} = \exp(W_t - t/2)$. Which of the following is not true?

- (a) $W_t^2 - t, t \geq 0$, is a Q -martingale.
- (b) $\exp(W_t - 3t/2), t \geq 0$, is a Q -martingale.
- (c) $f(W_t), t \geq 0$, is a (Q, \mathbb{F}) -semimartingale for $f \in C^2(\mathbb{R})$.

Solution 14.11 (a).

Exercise 14.12 Consider the Black–Scholes model. Then:

- (a) The price of a European put option increases if the strike is decreased.
- (b) The price of a European put option increases if the interest rate is increased.
- (c) None of the above.

Solution 14.12 (c).

Exercise 14.13 Let M be a local (P, \mathbb{F}) -martingale and A a process of finite variation. Then:

- (a) $[M, A] = 0$ P -a.s.
- (b) $[M, A] = 0$ P -a.s. if and only if both M and A are continuous.
- (c) $[M, A] = 0$ P -a.s. if A is continuous.

Solution 14.13 (c).

Exercise 14.14 Let $Z = (Z_t)_{t \in [0, T]}$ for some $T \in (0, \infty)$ be the density process of $Q \approx P$ on \mathcal{F}_T with respect to P . Which of the following is true?

- (a) $Z = Z_0 \mathcal{E}(L)$ for some continuous local (P, \mathbb{F}) -martingale $L = (L_t)_{t \in [0, T]}$ null at 0.
- (b) $Z = Z_0 \mathcal{E}(L)$ for some local (P, \mathbb{F}) -martingale $L = (L_t)_{t \in [0, T]}$ null at 0.
- (c) $Z = Z_0 \mathcal{E}(L)$ for some local (Q, \mathbb{F}) -martingale $L = (L_t)_{t \in [0, T]}$ null at 0.

Solution 14.14 (b).

Exercise 14.15 Which of the following statements about W is not true?

- (a) W has infinite 1-variation P -a.s.
- (b) W is a (P, \mathbb{F}) -semimartingale.
- (c) W is the unique continuous process with zero mean and normally distributed increments.

Solution 14.15 (c).

Exercise 14.16 Let X be a (P, \mathbb{F}) -semimartingale and $Q \stackrel{\text{loc}}{\approx} P$. Then:

- (a) $f(X)$ is a (Q, \mathbb{F}) -semimartingale for any C^∞ -function f .
- (b) X is a (Q, \mathbb{F}) -martingale.
- (c) $f(X)$ is a (P, \mathbb{F}) -semimartingale for any measurable function f .

Solution 14.16 (a).