Mathematical Foundations for Finance

Exercise Sheet 1

*The first three exercises of this sheet ask to establish some of the claims given in the lecture. The rest of the exercises contain material which is fundamental for the course and **assumed to be known**. Please hand in your solutions by 12:00 on Wednesday, September 28 via the course homepage.*

Exercise 1.1 Recall that

$$\mathcal{G} := \{g = G_T(\vartheta) : \vartheta \in \Theta\}$$

is the set of all possible final (time-T) wealth amounts one can generate from zero initial capital, and that

$$L^0_+ := \{X : X \geqslant 0\}$$

is the set of (equivalence classes of) random variables that are (P-a.s.) nonnegative. Recall also that for two sets A and B, $A - B := \{a - b : a \in A, b \in B\}$ denotes the set of ordered differences between elements of A and B.

Prove that absence of arbitrage, i.e. $\mathcal{G} \cap L^0_+ = \{0\}$, is equivalent to

$$(\mathcal{G} - L^0_+) \cap L^0_+ = \{0\}.$$

Solution 1.1 By definition, no arbitrage means that $\mathcal{G} \cap L^0_+ = \{0\}$ (if one starts from zero initial capital and has zero probability of losing money, the (final) gains must be P-a.s. zero).

Assume first absence of arbitrage, so that $\mathcal{G} \cap L^0_+ = \{0\}$. Take an arbitrary element of $(\mathcal{G} - L^0_+) \cap L^0_+$. It can be expressed as $g - X$ for some $g \in \mathcal{G}$ and $X \in L^0_+$. Since $g - X \in L^0_+$, it is nonnegative almost surely, and hence so is $g = g - X + X$. Hence, $g \in \mathcal{G} \cap L^0_+ = \{0\}$, and so $g = 0$. It follows that $-X = g - X \in L^0_+$, and thus $X = 0$. We have shown that $g - X = 0$, and so any element of $(\mathcal{G} - L^0_+) \cap L^0_+$ must be zero. Since clearly $0 \in (\mathcal{G} - L^0_+) \cap L^0_+$, we have that $(\mathcal{G} - L^0_+) \cap L^0_+ = \{0\}$, as required.

Conversely, suppose that $(\mathcal{G} - L^0_+) \cap L^0_+ = \{0\}$. Since $0 \in L^0_+$, we have $\mathcal{G} \subseteq \mathcal{G} - L^0_+$, and hence $\mathcal{G} \cap L^0_+ \subseteq (\mathcal{G} - L^0_+) \cap L^0_+ = \{0\}$. Since clearly $0 \in \mathcal{G} \cap L^0_+$, it follows that $\mathcal{G} \cap L^0_+ = \{0\}$, completing the proof.

Exercise 1.2 In the lectures, we saw that a sufficient condition for guaranteeing the absence of arbitrage is

$$E[g] = 0, \qquad \forall g \in \mathcal{G}, \tag{1}$$

i.e. that any possible wealth amount one can generate from zero initial capital has expectation zero.

(a) Prove that for a nonnegative random variable $X \in L^0_+$,

$$E[X] = 0 \qquad \Longrightarrow \qquad X = 0 \quad P\text{-a.s.}$$

(b) Using part (a), explain why (1) is indeed a sufficient condition for the absence of arbitrage.

Solution 1.2

(a) Since $X \in L^0_+$, then $1 = P[X \geqslant 0] = P[X = 0] + P[X > 0]$. It thus suffices to show that $P[X > 0] = 0$. Since $\{X \geqslant \frac{1}{n}\} \uparrow \{X > 0\}$, the monotone convergence theorem implies that

$$P[X > 0] = \lim_{n\to\infty} P[X \geqslant \tfrac{1}{n}].$$

Again using $X \in L^0_+$, we can apply Markov's inequality to get

$$P[X \geqslant \tfrac{1}{n}] \leqslant nE[X] = 0,$$

and hence $P[X \geqslant \frac{1}{n}] = 0$. It follows immediately that $P[X > 0] = 0$, as required.

(b) Assuming (1), we have for any $g \in \mathcal{G} \cap L^0_+$ that $E[g] = 0$. Since $g \in L^0_+$, part (a) gives that $g = 0$, and hence $\mathcal{G} \cap L^0_+ = \{0\}$.

Exercise 1.3 Consider a financial contract where the seller of the contract is obligated to pay the (random) amount H at expiry (time T) to the buyer. Define the set

$$\mathcal{Q} := \{\text{probability measure } Q \approx P : E_Q[g] \leqslant 0, \quad \forall g \in \mathcal{G}\}.$$

Recall that the fundamental theorem of asset pricing asserts that this set is nonempty.

(a) The *superreplication price* of the contract is given by

$$\pi(H) := \inf\{x \in \mathbb{R} : \exists \vartheta \text{ with } V_T(x, \vartheta) \geqslant H \ P\text{-a.s.}\}.$$

Prove that

$$\pi(H) \geqslant \sup\{E_Q[H] : Q \in \mathcal{Q}\}.$$

(Note: it is in fact true that $\pi(H) = \sup\{E_Q[H] : Q \in \mathcal{Q}\}$, but the proof of "$\leqslant$" relies on separation theorems and is therefore beyond the scope of this exercise. This equality is known as the *hedging duality*.)

(b) For $x \in \mathbb{R}$, define the set

$$\begin{aligned}\mathcal{C}(x) &= \{\mathcal{F}_T\text{-measurable } f : \exists \vartheta \in \Theta \text{ with } V_T(x,\vartheta) \geqslant f \ P\text{-a.s.}\} \\ &= \{\mathcal{F}_T\text{-measurable } f : \exists g \in \mathcal{G} \text{ with } f \leqslant x + g\} \\ &= x + \mathcal{G} - L^0_+(\mathcal{F}_T)\end{aligned}$$

to be the collection of all possible time-T payoffs that are affordable from initial capital x via trading in the financial market.

Prove that

$$f \in \mathcal{C}(x) \qquad \Longrightarrow \qquad \sup_{Q \in \mathcal{Q}} E_Q[f] \leqslant x.$$

(Note: in fact, these two statements are equivalent, but the implication "$\Leftarrow$" is considerably more difficult.)

Solution 1.3

(a) Fix $Q \in \mathcal{Q}$ and take $x \in \mathbb{R}$ with $H \leqslant V_T(x,\vartheta) = x + G_T(\vartheta) = x + g$ P-a.s. for some $\vartheta \in \Theta$ (respectively $g \in \mathcal{G}$). Since $Q \approx P$, we also have $H \leqslant x + g$ Q-a.s., and $E_Q[g] \leqslant 0$. So $E_Q[H] \leqslant x$, and as Q and x were arbitrary, we get

$$\sup_{Q \in \mathcal{Q}} E_Q[H] \leqslant \inf\{x \in \mathbb{R} : \exists \vartheta \text{ with } V_T(x,\vartheta) \geqslant H \ P\text{-a.s.}\},$$

and hence the assertion.

(b) Suppose that $f \in \mathcal{C}(x)$ and take $Q \in \mathcal{Q}$. By definition of $\mathcal{C}(x)$, there exists ϑ such that $V_T(x,\vartheta) \geqslant f$ P-a.s. Since $Q \approx P$, then also $V_T(x,\vartheta) \geqslant f$ Q-a.s. Writing $V_T(x,\vartheta) = x + G_T(\vartheta)$ and using that $E_Q[G_T(\vartheta)] \leqslant 0$, we get

$$E_Q[f] \leqslant E_Q[V_T(x,\vartheta)] = x + E_Q[G_T(\vartheta)] \leqslant x.$$

Taking the supremum over all $Q \in \mathcal{Q}$ yields the claim.

An alternative proof is as follows:

Take $f \in \mathcal{C}(x)$. Then there exists $\vartheta \in \Theta$ with $V_T(x,\vartheta) \geqslant f$ P-a.s. This implies by the definition of the superreplication price that $\pi(f) \leqslant x$, and then (a) implies that

$$\sup_{Q \in \mathcal{Q}} E_Q[f] \leqslant \pi(f) \leqslant x.$$

Exercise 1.4 Let Ω be a set.

(a) Suppose that $\{\mathcal{F}_j\}_{j \in J}$ is a nonempty family of σ-fields on Ω. Prove that the intersection $\bigcap_{j \in J} \mathcal{F}_j$ is a σ-field on Ω.

(b) Let $\mathcal{A}$ be a family of subsets of Ω. Show that there is a (clearly unique) minimal σ-field $\sigma(\mathcal{A})$ containing $\mathcal{A}$. Here minimality is with respect to inclusion: if $\mathcal{F}$ is a σ-field with $\mathcal{A} \subseteq \mathcal{F}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.

(c) Give an example of two σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega = \{1, 2, 3\}$ whose union $\mathcal{F}_1 \cup \mathcal{F}_2$ is *not* a σ-field.

Solution 1.4

(a) We check the requirements for a σ-field:

- $\Omega \in \bigcap_{j \in J} \mathcal{F}_j$ because $\Omega \in \mathcal{F}_j$ for all $j \in J$.
- If $A \in \bigcap_{j \in J} \mathcal{F}_j$, then $A \in \mathcal{F}_j$ for all $j \in J$, and hence $A^c \in \mathcal{F}_j$ for all $j \in J$, so that $A^c \in \bigcap_{j \in J} \mathcal{F}_j$.
- If $A_n \in \bigcap_{j \in J} \mathcal{F}_j$, $n \in \mathbb{N}$, then $A_n \in \mathcal{F}_j$ for all $j \in J$, and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_j$ for all $j \in J$, so that $\bigcup_{n=1}^{\infty} A_n \in \bigcap_{j \in J} \mathcal{F}_j$.

Thus, $\bigcap_{j \in J} \mathcal{F}_j$ is a σ-field.

(b) Define

$$\sigma(\mathcal{A}) := \bigcap_{\substack{\mathcal{F}\ \sigma\text{-field} \\ \mathcal{F} \supseteq \mathcal{A}}} \mathcal{F}.$$

Note that the above intersection is over a non-empty family of σ-fields, since the power set on Ω is a σ-field that contains $\mathcal{A}$. By part (a), $\sigma(\mathcal{A})$ is a σ-field, and of course $\sigma(\mathcal{A}) \supseteq \mathcal{A}$. The uniqueness of such a σ-field follows immediately from the construction of $\sigma(\mathcal{A})$.

(c) Consider the σ-fields

$$\mathcal{F}_1 := \sigma(\{\{1\}\}) = \{\varnothing, \{1, 2, 3\}, \{1\}, \{2, 3\}\},$$
$$\mathcal{F}_2 := \sigma(\{\{2\}\}) = \{\varnothing, \{1, 2, 3\}, \{2\}, \{1, 3\}\}.$$

We see that

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\varnothing, \{1, 2, 3\}, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}.$$

This is not a σ-field, since $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

Exercise 1.5 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Then the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E\,[X \mathbb{1}_A] = E[Z \mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called the *conditional expectation of* X *given* $\mathcal{G}$ and is denoted by $E[X \mid \mathcal{G}]$. (This is the formal definition of conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.)

(a) Show that if X is $\mathcal{G}$-measurable, then $E[X \mid \mathcal{G}] = X$ P-a.s.

(b) Show that $E[E[X \mid \mathcal{G}]] = E[X]$.

(c) Show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$ (that is, if $\mathcal{G}$ is P-trivial), then $E[X \mid \mathcal{G}] = E[X]$ P-a.s.

(d) Consider an integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and constants $a, b \in \mathbb{R}$. Show that $E[aX + bY \mid \mathcal{G}] = aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e., there exists a collection $(A_i)_{i=1,\dots,n}$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \dots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \dots, n$. Show that

$$E[X \mid \mathcal{G}] = \sum_{i=1}^{n} E[X \mid A_i] \mathbb{1}_{A_i} \qquad P\text{-a.s.}$$

This says that the conditional expectation of a random variable given a finitely generated σ-algebra is a *piecewise constant* function with the constants given by the elementary conditional expectations given the sets of the generating partition.

[*This is a very useful property when one conditions on a finitely generated σ-field, as for instance in the multinomial model.*]

Hint 1: Recall that $E[X \mid A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$ and try to write X as a sum of random variables each of which only takes non-zero values on a single A_i.
Hint 2: Check that any set $A \in \mathcal{G}$ has the form $\cup_{j\in J} A_j$ for some $J \subseteq \{1, \dots, n\}$.

Solution 1.5

(a) X is $\mathcal{G}$-measurable and integrable by assumption, so the first requirement in the definition of conditional expectation is satisfied for $Z = X$. Moreover, we clearly have that $E[X\mathbb{1}_A] = E[X\mathbb{1}_A]$ for all $A \in \mathcal{G}$, hence $E[X \mid \mathcal{G}] = X$ P-a.s.

(b) In the definition of the conditional expectation, set $A = \Omega$. Then we obtain that $E[E[X \mid \mathcal{G}]] = E[E[X \mid \mathcal{G}]\mathbb{1}_\Omega] = E[X\mathbb{1}_\Omega] = E[X]$.

(c) Since $|E[X]| \leq E[|X|]$ by Jensen's inequality and $E[|X|] < \infty$ since X is integrable by assumption, we have that $E[X]$ is integrable as well. $E[X]$ is also trivially $\mathcal{G}$-measurable since it is a constant random variable. Moreover, in this setting, $A \in \mathcal{G}$ only if $P[A] = 0$ or $P[A] = 1$. Noting that

$$\begin{aligned} E[X\mathbb{1}_A] &= 0 = E[E[X]\,\mathbb{1}_A], && \forall\ A \in \mathcal{G} \text{ such that } P[A] = 0,\\ E[X\mathbb{1}_A] &= E[X] = E[E[X]\,\mathbb{1}_A], && \forall\ A \in \mathcal{G} \text{ such that } P[A] = 1, \end{aligned}$$

we obtain $E[X\,|\,\mathcal{G}] = E[X]$ P-a.s.

(d) By the definition of the conditional expectation, we have that $E[X\,|\,\mathcal{G}]$ and $E[Y\,|\,\mathcal{G}]$ are $\mathcal{G}$-measurable and integrable; hence the same holds for $aE[X\,|\,\mathcal{G}] + bE[Y\,|\,\mathcal{G}]$. Choosing some $A \in \mathcal{G}$, we can compute that

$$\begin{aligned} E\Big[(aE[X\,|\,\mathcal{G}] + bE[Y\,|\,\mathcal{G}])\mathbb{1}_A\Big] &= aE\Big[E[X\,|\,\mathcal{G}]\,\mathbb{1}_A\Big] + bE\Big[E[Y\,|\,\mathcal{G}]\,\mathbb{1}_A\Big]\\ &= aE[X\mathbb{1}_A] + bE[Y\mathbb{1}_A] = E[(aX+bY)\mathbb{1}_A], \end{aligned}$$

where the first equality uses the linearity of the (classical) expectation and the second uses the definition of $E[X\,|\,\mathcal{G}]$ and $E[Y\,|\,\mathcal{G}]$. Because $A \in \mathcal{G}$ was arbitrary, this shows that $E[aX + bY\,|\,\mathcal{G}] = aE[X\,|\,\mathcal{G}] + bE[Y\,|\,\mathcal{G}]$ P-a.s.

(e) First recall that $E[X\,|\,A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$. Using that

$$X = X\mathbb{1}_\Omega = X\mathbb{1}_{\cup_{i=1}^n A_i} = X\sum_{i=1}^n \mathbb{1}_{A_i} = \sum_{i=1}^n X\mathbb{1}_{A_i},$$

where the third equality holds because A_i are pairwise disjoint, we get by part (d) that

$$E[X\,|\,\mathcal{G}] = \sum_{i=1}^n E[X\mathbb{1}_{A_i}\,|\,\mathcal{G}] \quad P\text{-a.s.},$$

and hence we only have to show that $E[X\mathbb{1}_{A_i}\,|\,\mathcal{G}] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ P-a.s. for each $i \in \{1,\dots,n\}$.

Since $A_i \in \mathcal{G}$ and $E[X\,|\,A_i] = E[X\mathbb{1}_{A_i}]/P[A_i] \in \mathbb{R}$, we already know that $E[X\,|\,A_i]\,\mathbb{1}_{A_i}$ is $\mathcal{G}$-measurable and integrable. One can verify that the family of sets $A = \bigcup_{j\in J} A_j$ for $J \in 2^{\{1,\dots,n\}}$ (the power set of $\{1,\dots,n\}$) forms a σ-field. Let us denote this σ-field by $\tilde{\mathcal{G}}$. Since we clearly have $A_i \in \tilde{\mathcal{G}}$ for all $i \in \{1,\dots,n\}$, we get that $\tilde{\mathcal{G}} \supseteq \mathcal{G}$, which for any $A \in \mathcal{G}$ implies that $A = \bigcup_{j\in J} A_j$ for some $J \subseteq \{1,\dots,n\}$. For any such $A \in \mathcal{G}$, we have that

$$\mathbb{1}_{A_i}\mathbb{1}_A = \begin{cases} \mathbb{1}_{A_i} & \text{if } i \in J,\\ 0, & \text{else.} \end{cases}$$

Hence we can then compute

$$E\left[\left(\frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}\right)\mathbb{1}_A\right] = \begin{cases} E[X\mathbb{1}_{A_i}]\,\frac{P[A_i]}{P[A_i]} = E[X\mathbb{1}_{A_i}] & \text{if } i \in J,\\ 0 & \text{else.} \end{cases}$$

On the other hand, we have that

$$E\left[X\mathbb{1}_{A_i}\mathbb{1}_A\right] = \begin{cases} E\left[X\mathbb{1}_{A_i}\right] & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

This shows that $E\left[X\mathbb{1}_{A_i}\,|\,\mathcal{G}\right] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ P-a.s. and concludes the proof.