# Mathematical Foundations for Finance

## Exercise Sheet 2

*Please hand in your solutions by 12:00 on Wednesday, October 5 via the course homepage.*

**Exercise 2.1** Let $(\Omega, \mathcal{F}, P)$ be the probability space with $\Omega = \{UU, UD, DD, DU\}$, $\mathcal{F} = 2^\Omega$, and $P$ defined by $P[\omega] = 1/4$ for all $\omega \in \Omega$ (so $P$ is the *uniform* probability measure on $\Omega$). Consider the random variables $Y_1, Y_2 \colon \Omega \to \mathbb{R}$ that are given by $Y_1(UU) = Y_1(UD) = 2$, $Y_1(DD) = Y_1(DU) = 1/2$, $Y_2(UU) = Y_2(DU) = 2$, and $Y_2(DD) = Y_2(UD) = 1/2$. Define the process $X = (X_k)_{k=0,1,2}$ by $X_0 = 8$, and for $k = 1, 2$, $X_k = X_0 \prod_{i=1}^k Y_i$.

(a) Draw a tree to illustrate the possible evolutions of the process $X$ from time 0 to time 2, and label the corresponding transition probabilities and probabilities.

(b) For $k = 0, 1, 2,$, write down the $\sigma$-fields (i.e. give all their sets) defined by $\mathcal{F}_k = \sigma(X_i : 0 \leqslant i \leqslant k)$ and $\mathcal{G}_k = \sigma(X_k)$.

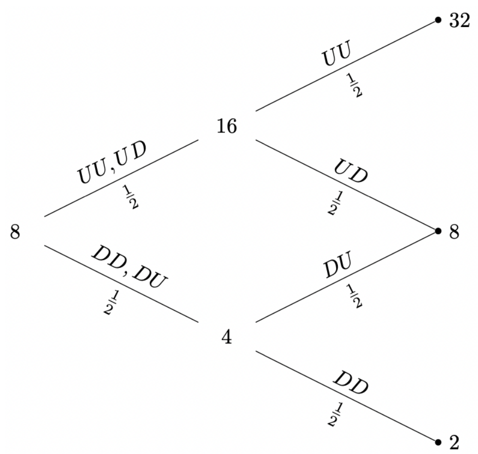
(c) Consider the collections of $\sigma$-fields $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ and $\mathbb{G} = (\mathcal{G}_k)_{k=0,1,2}$. Do these form filtrations on $(\Omega, \mathcal{F})$? Why or why not?

(d) If they are indeed filtrations, is $X$ adapted to $\mathbb{F}$ or $\mathbb{G}$?

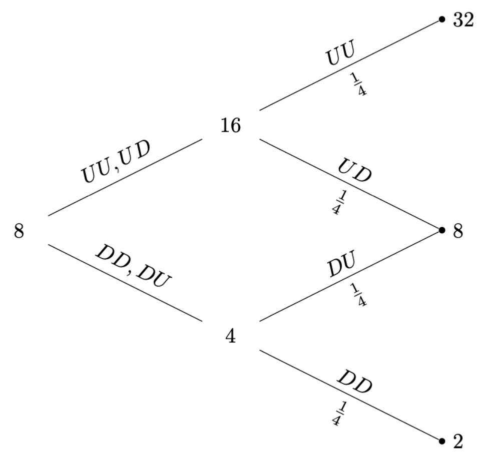
(e) Give financial interpretations of $X$, $\mathbb{F}$ and $\mathbb{G}$.

### Solution 2.1

(a) The tree with the transition probabilities labelled is drawn below.



Next, this is the tree with the probabilities labelled.




(b) Since $X_0 = 8$ is a constant, then $\mathcal{F}_0 = \mathcal{G}_0 = \sigma(X_0) = \{\varnothing, \Omega\}$. We thus also have $\mathcal{F}_1 = \sigma(X_0, X_1) = \sigma(X_1) = \mathcal{G}_1$. Moreover, because $X_1$ is either 4 or 16, and $X_1^{-1}(4) = \{DD, DU\}$ and $X_1^{-1}(16) = \{UU, UD\}$, we have

$$\sigma(X_1) = \{\varnothing, \Omega, \{DD, DU\}, \{UU, UD\}\},$$

(since the right hand side above is a $\sigma$-field). By the same reasoning, since $X_2$ is either 2, 8, or 32, and $X_2^{-1}(2) = \{DD\}$, $X_2^{-1}(8) = \{DU, UD\}$, and $X_2^{-1}(32) = \{UU\}$, we have

$$\begin{aligned}\mathcal{G}_2 = \sigma(X_2) &= \sigma\left(\{DD\}, \{DU, UD\}, \{UU\}\right)\\ &= \{\varnothing, \Omega, \{DD\}, \{DU, UD\}, \{UU\}, \{DD, DU, UD\}, \{DU, UD, UU\},\\ &\quad \{DD, UU\}\}.\end{aligned}$$

Since $\{DD, DU\}, \{UU, UD\} \in \sigma(X_1)$ and $\{DU, UD\}, \{DD, UU\} \in \sigma(X_2)$, then by taking intersections we see that $\{DU\}, \{UD\} \in \sigma(X_1, X_2)$. Since also $\{DD\}, \{UU\} \in \sigma(X_2)$, we get

$$\mathcal{F}_2 = \sigma(X_0, X_1, X_2) = 2^{\Omega} = \mathcal{F}.$$

(c) By construction, $\mathbb{F}$ is a filtration on $(\Omega, \mathcal{F})$ (of course, this can also be checked directly). Since $\{DD, DU\} \in \mathcal{G}_1 \backslash \mathcal{G}_2$, then $\mathcal{G}_1 \subsetneq \mathcal{G}_2$, and hence $\mathbb{G}$ is not a filtration.

(d) By construction, $X$ is adapted to $\mathbb{F}$. Also, $X_k$ is of course $\mathcal{G}_k$-measurable for each $k = 0, 1, 2$, but $\mathbb{G}$ is not a filtration.

(e) The process $X$ can be interpreted as the price of a stock that is worth 8 at time zero, and at each period changes by a factor of either 1/2 or 2.

The filtration $\mathbb{F}$ can be thought of as the *cumulative* information that the stock price evolution provides us with over time.

The collection $\mathbb{G}$ can be though of as the information we know by only observing the present stock price, but not the past stock prices.

**Exercise 2.2** *(Multiplicative model)* Consider a probability space $(\Omega, \mathcal{F}, P)$. Fix a finite time horizon $T \in \mathbb{N}$, and let $r_1, \dots, r_T > -1$ and $Y_1, \dots, Y_T > 0$ be random variables. For $k = 0, \dots, T$, define

$$\widetilde{S}_k^0 := \prod_{j=1}^{k} (1 + r_j), \qquad \widetilde{S}_k^1 := S_0^1 \prod_{j=1}^{k} Y_j,$$

where $S_0^1 > 0$ is some constant.

(a) Consider the filtration $\mathbb{F}' = (\mathcal{F}'_k)_{k=0,\dots,T}$ generated by $Y = (Y_k)_{k=1,\dots,T}$ and $r = (r_k)_{k=1,\dots,T}$, so that

$$\mathcal{F}'_0 = \{\varnothing, \Omega\},$$
$$\mathcal{F}'_k = \sigma(Y_1, \dots, Y_k, r_1, \dots, r_k), \qquad k = 1, \dots, T.$$

Show that if $r$ is $\mathbb{F}'$-predictable, then $\mathcal{F}'_k = \mathcal{F}_k := \sigma(\widetilde{S}_0^1, \widetilde{S}_1^1, \dots, \widetilde{S}_k^1)$ for all $k = 0, \dots, T$.

(b) Recall that a strategy $\varphi = (\varphi^0, \vartheta)$ is *self-financing* if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing does not depend on discounting. That is, if $D = (D_k)_{k=0,\dots,T}$ is any positive adapted process and $\bar{S}_k^i := S_k^i D_k$ for each $k = 0, \dots, T$ and $i = 0, 1$, then the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\bar{C}(\varphi)$, determined by

$$\Delta \bar{C}_{k+1}(\varphi) := (\varphi_{k+1}^0 - \varphi_k^0)\bar{S}_k^0 + (\vartheta_{k+1} - \vartheta_k)\bar{S}_k^1,$$

is constant over time.

(c) Show that the notion of self-financing is numéraire-invariant, i.e. it does not matter if the discounted price processes are defined as $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$, or $\bar{S}^0 := \widetilde{S}^0/\widetilde{S}^1$ and $\bar{S}^1 := \widetilde{S}^1/\widetilde{S}^1$.

**Solution 2.2**

(a) The proof is by induction on $k$. Since $\widetilde{S}_0^1 = S_0^1$ is constant, then

$$\mathcal{F}_0 = \sigma(\widetilde{S}_0^1) = \{\varnothing, \Omega\} = \mathcal{F}'_0.$$

Now assume that $\mathcal{F}_k = \mathcal{F}'_k$ for some $k \geqslant 0$. We need to show that $\mathcal{F}_{k+1} = \mathcal{F}'_{k+1}$. To this end, note that

$$\widetilde{S}_{k+1}^1 = \widetilde{S}_k^1 Y_{k+1},$$

which is $\mathcal{F}'_{k+1}$-measurable, since $\widetilde{S}^1_k$ (because $\mathcal{F}'_k = \mathcal{F}_k$) and $Y_{k+1}$ are. So since $\widetilde{S}^1_j$ is $\mathcal{F}'_{k+1}$-measurable for all $0 \leqslant j \leqslant k+1$, we have $\mathcal{F}_{k+1} \subseteq \mathcal{F}'_{k+1}$. Conversely, writing

$$Y_{k+1} = \frac{\widetilde{S}^1_{k+1}}{\widetilde{S}^1_k}$$

(which is well-defined since $\widetilde{S}^1_k > 0$), we see that $Y_{k+1}$ is $\mathcal{F}_{k+1}$-measurable. Also, since $r$ is $\mathbb{F}'$-predictable, then $r_{k+1}$ is $\mathcal{F}_k$-measurable (because $\mathcal{F}_k = \mathcal{F}'_k$). By the same reasoning as above, since $Y_j$ and $r_j$ are $\mathcal{F}_k$-measurable for all $0 \leqslant j \leqslant k+1$, we get $\mathcal{F}'_{k+1} \subseteq \mathcal{F}_{k+1}$. Hence, $\mathcal{F}'_{k+1} = \mathcal{F}_{k+1}$. By induction, this completes the proof.

(b) By applying the definition of the incremental cost $k$ times, we get

$$C_k(\varphi) = \Delta C_k(\varphi) + C_{k-1}(\varphi) = C_0(\varphi) + \sum_{j=1}^{k} \Delta C_j(\varphi).$$

It follows that the cost process $C(\varphi)$ is constant over time (and equal to the initial investment $\varphi^0_0$ in the bank account) if and only if we have that $\Delta C_k(\varphi) = 0$ for all $k$. By the definition of $\Delta C_k(\varphi)$, this equality reads

$$(\varphi^0_k - \varphi^0_{k-1}) S^0_{k-1} + (\vartheta_k - \vartheta_{k-1}) S^1_{k-1} = 0, \qquad \forall k.$$

Multiplying both sides of the equation by $D_{k-1}$, we obtain the same condition for the prices $\bar{S}$:

$$\Delta \bar{C}_k(\varphi) = (\varphi^0_k - \varphi^0_{k-1}) \bar{S}^0_{k-1} + (\vartheta_k - \vartheta_{k-1}) \bar{S}^1_{k-1} = 0.$$

Since we also have the identity

$$\bar{C}_k(\varphi) = \bar{C}_0(\varphi) + \sum_{j=1}^{k} \Delta \bar{C}_j(\varphi),$$

it follows that the undiscounted cost process $\overline{C}(\phi)$ is constant over time. The other direction is established in the same way, thus completing the proof.

(c) This follows immediately from part (b) by first setting $D = \widetilde{S}^1 / \widetilde{S}^0$ and then $D = \widetilde{S}^0 / \widetilde{S}^1$.

**Exercise 2.3** Consider a probability space $(\Omega, \mathcal{F}, P)$. A $\sigma$-field $\mathcal{F}_0 \subseteq \mathcal{F}$ is said to be $P$-*trivial* if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{F}_0$. Prove that $\mathcal{F}_0$ is $P$-trivial if and only if every $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$ is $P$-a.s. constant.

*Note: by using the fact that $X = (X^1, \ldots, X^n) : \Omega \to \mathbb{R}^n$ is a random variable if and only if each of its components $X^i$ is a random variable, we can extend the above result to dimension $n$.*

**Solution 2.3** Suppose first that $\mathcal{F}_0$ is $P$-trivial, and consider an $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$. We then have that for all $a \in \mathbb{R}$, $\{X \leqslant a\} \in \mathcal{F}_0$, and thus $P[X \leqslant a] \in \{0, 1\}$. Define

$$c := \inf\{a \in \mathbb{R} : P[X \leqslant a] = 1\}.$$

Since $\{X \leqslant n\} \uparrow \{X \in \mathbb{R}\}$, then $P[X \leqslant n] \uparrow P[X \in \mathbb{R}] = 1$, and so the above infimum is over a nonempty set (i.e. $c \neq \infty$). If $c = -\infty$, then $P[X \leqslant -n] = 1$ for all $n \in \mathbb{N}$, and since $\{X \leqslant -n\} \downarrow \varnothing$, we have $1 = \lim_{n\to\infty} P[X \leqslant -n] = P[\varnothing] = 0$, a contradiction. Hence $c \in \mathbb{R}$. By the definition of the infimum, we have that for all $n \in \mathbb{N}$, $P[X \leqslant c + \frac{1}{n}] = 1$ and $P[X \leqslant c - \frac{1}{n}] = 0$. Since $\{X \leqslant c + \frac{1}{n}\} \downarrow \{X \leqslant c\}$ and $\{X \leqslant c - \frac{1}{n}\} \uparrow \{X < c\}$, we get $P[X \leqslant c] = \lim_{n\to\infty} P[X \leqslant c + \frac{1}{n}] = 1$ and $P[X < c] = \lim_{n\to\infty} P[X \leqslant c - \frac{1}{n}] = 0$. Hence,

$$P[X = c] = P[X \leqslant c] - P[X < c] = 1,$$

so that $X = c$ $P$-a.s.

Conversely, suppose that every $\mathcal{F}_0$-measurable random variable is $P$-a.s. constant, and take $A \in \mathcal{F}_0$. Then

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c \end{cases}$$

is an $\mathcal{F}_0$-measurable random variable, and hence must be $P$-a.s. constant. It follows immediately that either $P[\mathbb{1}_A = 1] = P[A] = 1$ or $P[\mathbb{1}_A = 0] = P[A^c] = 1$, so that $P[A] \in \{0, 1\}$. This completes the proof.

**Exercise 2.4** Consider a probability space $(\Omega, \mathcal{F}, P)$, and let $X : \Omega \to \mathbb{R}$ be a random variable.

(a) Prove that $4X - 7$ is a random variable.

(b) Prove that for any continuous function $h : \mathbb{R} \to \mathbb{R}$, the composition $h(X) : \Omega \to \mathbb{R}$ is a random variable.

*You may use the following fact:*

*A function $X : \Omega \to \mathbb{R}$ is a random variable if and only if one (and hence all) of the following conditions hold:*

- $\{X \leqslant c\} \in \mathcal{F}$ *for all* $c \in \mathbb{R}$,
- $\{X < c\} \in \mathcal{F}$ *for all* $c \in \mathbb{R}$,
- $\{X \geqslant c\} \in \mathcal{F}$ *for all* $c \in \mathbb{R}$,
- $\{X > c\} \in \mathcal{F}$ *for all* $c \in \mathbb{R}$.

**Solution 2.4**

(a) This follows immediately from part (b) by considering the continuous function $h(x) = 4x - 7$.

Alternatively, we can prove this directly by using the hint. For example, we have that for any $c \in \mathbb{R}$,

$$\{4X - 7 \leqslant c\} = \{X \leqslant \tfrac{c+7}{4}\} \in \mathcal{F},$$

and the conclusion follows.

(b) We show that we have $\{h(X) < c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$. Using the identity $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, we write

$$\{h(X) < c\} = \Big(h(X)\Big)^{-1}(-\infty, c) = X^{-1}\Big(h^{-1}(-\infty, c)\Big).$$

Since $(-\infty, c)$ is open and $h$ is continuous, then $h^{-1}(-\infty, c)$ is also open, and hence $h^{-1}(-\infty, c) \in \mathcal{B}(\mathbb{R})$. Since $X$ is a random variable, we have $X^{-1}(h^{-1}(-\infty, c)) \in \mathcal{F}$, completing the proof.