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Mathematical Foundations for Finance

Exercise Sheet 3

Please hand in your solutions by 12:00 on Wednesday, October 12 via the course homepage.

Exercise 3.1 *(Stopping times)* Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_k)_{k=0,\dots,T}$. Let $\tau, \sigma : \Omega \to \mathbb{N} \cup \{\infty\}$ be stopping times.

(a) Show that $\tau \wedge \sigma := \min\{\tau, \sigma\}$ is a stopping time.

(b) Show that $\tau \vee \sigma := \max\{\tau, \sigma\}$ is a stopping time.

(c) Show that a function $\rho : \Omega \to \mathbb{N} \cup \{\infty\}$ is an $\mathbb{F}$-stopping time if and only if $\{\rho = k\} \in \mathcal{F}_k$ for all $k = 0, \dots, T$.

(d) Show that $\tau + \sigma$ is a stopping time.

(e) Suppose $\tau \geqslant \sigma$. Is $\tau - \sigma$ a stopping time?

(f) Suppose that $X = (X_k)_{k=0,\dots,T}$ is an adapted $\mathbb{R}^d$-valued process, and let $a \in \mathbb{R}$. Show that

$$\rho := \inf\{k : |X_k| \geqslant a\}$$

is a stopping time.

Show that ρ is still a stopping time if "$\geqslant$" is replaced by any of "$>$", "$\leqslant$" or "$<$".

Solution 3.1

(a) We have $\{\tau \wedge \sigma \leqslant k\} = \{\tau \leqslant k\} \cup \{\sigma \leqslant k\} \in \mathcal{F}_k$, since τ and σ are stopping times.

(b) We have $\{\tau \vee \sigma \leqslant k\} = \{\tau \leqslant k\} \cap \{\sigma \leqslant k\} \in \mathcal{F}_k$, since τ and σ are stopping times.

(c) If τ is a stopping time, then $\{\tau = k\} = \{\tau \leqslant k\} \backslash \{\tau \leqslant k-1\} \in \mathcal{F}_k$, as needed. For the converse, we note that $\{\tau \leqslant k\} = \cup_{j=0}^{k} \{\tau = j\} \in \mathcal{F}_k$, as required.

(d) We have

$$\{\tau + \sigma = k\} = \bigcup_{j=0}^{k} \{\tau = j\} \cap \{\sigma = k - j\}.$$

By part (c), we can conclude that $\{\tau + \sigma = k\} \in \mathcal{F}_k$. Then again by part (c), this implies that $\tau + \sigma$ is a stopping time.

(e) No. Keeping part (c) in mind, take stopping times $\tau \equiv 1$ and $\sigma : \Omega \to \{0, 1\}$, where $\{\sigma = 1\} \in \mathcal{F}_1 \backslash \mathcal{F}_0$. Then we have

$$\{\tau - \sigma = 0\} = \{\sigma = 1\} \notin \mathcal{F}_0,$$

so that $\tau - \sigma$ cannot be a stopping time.

(f) For $n = 0, \ldots, T$, we have

$$\{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| \geqslant a\}.$$

Since X is adapted, then for all $k = 0, \ldots, n$, we have

$$\{|X_k| \geqslant a\} \in \mathcal{F}_k \subseteq \mathcal{F}_n.$$

Since a σ-field is closed under finite unions, it follows that $\{\rho \leqslant n\} \in \mathcal{F}_n$, and hence ρ is a stopping time.

Similarly, when "$\geqslant$" is replaced by "$>$", "$\leqslant$" or "$<$", we have the following equalities, respectively:

$$\{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| > a\}, \qquad \{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| \leqslant a\},$$
$$\{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| < a\}.$$

By the same reasoning as above, we can conclude that ρ is still a stopping time in these cases.

Exercise 3.2 *(Trading strategies)* Fix a probability space $(\Omega, \mathcal{F}, P)$ and a finite time horizon $T \geqslant 2$. Consider a market (S^0, S^1) consisting of a bank account and a stock, respectively. Assume that $S^0 \equiv 1$, $S^1_0 = 1$ and $S^1_k > 0$ for all $k = 1, \ldots, T$. Fix $0 < \ell < 1 < u$, and define the maps $\tau, \sigma : \Omega \to \mathbb{N} \cup \{\infty\}$ by

$$\tau := \inf\{k : S^1_k \leqslant \ell\} \wedge T,$$
$$\sigma := \inf\{k : S^1_k \geqslant u\} \wedge T.$$

We use here the standard convention $\inf \varnothing = +\infty$.

(a) Define the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,\ldots,T}$ on $(\Omega, \mathcal{F})$ by $\mathcal{F}_k := \sigma(S^1_i : 0 \leqslant i \leqslant k)$. Show that τ and σ are $\mathbb{F}$-stopping times.

(b) Define the process $\vartheta = (\vartheta_k)_{k=1,\ldots,T}$ by

$$\vartheta_k := \mathbb{1}_{\{\tau < k \leqslant \sigma\}}, \qquad k = 1, \ldots, T.$$

Show that ϑ is $\mathbb{F}$-predictable and $\vartheta_1 = 0$.

(c) Construct φ^0 such that the strategy $\varphi = (\varphi^0, \vartheta)$ is self-financing with $V_0(\varphi) = 0$, and derive a formula for the discounted value process $V(\varphi)$ involving only the discounted stock price S^1 and the stopping times τ and σ.

(d) Describe the trading strategy φ in words.

Solution 3.2

(a) Since S^1 is adapted to $\mathbb{F}$, we can use Exercise 3.1(f) to conclude that the random variables $\inf\{k : S^1_k \leqslant \ell\}$ and $\inf\{k : S^1_k \geqslant u\}$ are stopping times. Since a nonnegative integer (in our case, T) is a stopping time, we can use Exercise 3.1(a) to obtain the result.

(b) Since an indicator function is measurable if and only if the indicating set belongs to the σ-field, we need to show that $\{\tau < k \leqslant \sigma\} \in \mathcal{F}_{k-1}$ for each $k = 1, \dots, T$. To this end, we write

$$\{\tau < k \leqslant \sigma\} = \{\tau < k\} \cap \{\sigma \geqslant k\} = \{\tau \leqslant k-1\} \cap \{\sigma \leqslant k-1\}^c.$$

Since τ and σ are stopping times, it follows that the above set belongs to $\mathcal{F}_{k-1}$, completing the proof.

Finally, note that since $S^1_0 = 1$, we must have $\tau \geqslant 1$, so that

$$\vartheta_1 = \mathbb{1}_{\{\tau < 1 \leqslant \sigma\}} = \mathbb{1}_{\varnothing} = 0,$$

as required.

(c) Recall that a strategy $\varphi = (\phi^0, \vartheta)$ is *self-financing* if $C_k(\varphi) = C_0(\varphi)$ for all k. By definition, $C_k(\varphi) = V_k(\varphi) - G_k(\vartheta)$, and since $C_0(\varphi) = V_0(\varphi)$, we may rewrite this condition as $V_k(\varphi) = V_0(\varphi) + G_k(\vartheta)$ for all k.

For our setting, we compute

$$\begin{aligned} G_k(\vartheta) &= \sum_{j=1}^{k} \vartheta_j \Delta S^1_j = \sum_{j=1}^{k} \mathbb{1}_{\{\tau < j \leqslant \sigma\}} \Delta S^1_j \\ &= \sum_{j=1}^{k} \mathbb{1}_{\{\tau < j \leqslant \sigma\}} (S^1_j - S^1_{j-1}) = \sum_{j=k\wedge\tau+1}^{k\wedge\sigma} (S^1_j - S^1_{j-1}) \\ &= S^1_{k\wedge\sigma} - S^1_{k\wedge\tau}. \end{aligned}$$

By definition, $V_k(\varphi) = \varphi^0_k + \vartheta_k S^1_k = \varphi^0_k + \mathbb{1}_{\{\tau < k \leqslant \sigma\}} S^1_k$, and thus φ is self-financing with $V_0(\varphi) = 0$ if and only if $\varphi^0_0 = 0$ and

$$\varphi^0_k + \mathbb{1}_{\{\tau < k \leqslant \sigma\}} S^1_k = S^1_{k\wedge\sigma} - S^1_{k\wedge\tau}, \qquad \forall k = 1, \dots, T.$$

We thus take $\varphi^0_0 = 0$, and for all $k = 1, \dots, T$,

$$\begin{aligned} \varphi^0_k &= S^1_{k\wedge\sigma} - S^1_{k\wedge\tau} - \mathbb{1}_{\{\tau < k \leqslant \sigma\}} S^1_k \\ &= -S^1_\tau \mathbb{1}_{\{\tau < k \leqslant \sigma\}} + \mathbb{1}_{\{\sigma < k\}} (S^1_\sigma - S^1_\tau). \end{aligned}$$

Finally, since $C_k(\varphi) = C_0(\varphi) = 0$ for all k, we have

$$V_k(\varphi) = G_k(\vartheta) = S^1_{k\wedge\sigma} - S^1_{k\wedge\tau},$$

so that

$$V(\varphi) = (S^1)^\sigma - (S^1)^\tau.$$

(d) This strategy can be described as a "buy low and sell high" strategy. When the discounted price of the stock falls below ℓ, one borrows money to buy one share of the stock. As soon as the discounted price of the stock climbs above u, one sells the share, pays back the loan and stores the remaining money in the bank account.

Exercise 3.3 *(The martingale property)* Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_k)_{k=0,\dots,T}$. Let $X = (X_k)_{k=0,\dots,T}$ be a real-valued, adapted and integrable process. Show that the following conditions are equivalent:

(1) X is a martingale,

(2) $E[X_{k+1} \mid \mathcal{F}_k] = X_k$ for all $k = 0, \dots, T-1$,

(3) $E[X_{k+1} - X_k \mid \mathcal{F}_k] = 0$ for all $k = 0, \dots, T-1$.

Solution 3.3 We prove (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2): This follows immediately from the definition of a martingale.

(2) $\Rightarrow$ (1): Fix $0 \leqslant m < n \leqslant T$. We need to show

$$E[X_n \mid \mathcal{F}_m] = X_m.$$

By the *tower law* of conditional expectations (if $\mathcal{G} \subseteq \mathcal{H}$ are sub-σ-fields and X is an integrable random variable, then $E[E[X \mid \mathcal{H}] \mid \mathcal{G}] = E[X \mid \mathcal{G}]$) we can write

$$E[X_n \mid \mathcal{F}_m] = E\Big[E[X_n \mid \mathcal{F}_{n-1}] \mid \mathcal{F}_m\Big].$$

We are given that $E[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$, and so we have

$$E[X_n \mid \mathcal{F}_m] = E[X_{n-1} \mid \mathcal{F}_m].$$

We can similarly show that $E[X_{n-1} \mid \mathcal{F}_m] = E[X_{n-2} \mid \mathcal{F}_m]$ (if $n-1 > m$), and we repeat this argument until we get

$$E[X_n \mid \mathcal{F}_m] = E[X_m \mid \mathcal{F}_m].$$

Since X_m is $\mathcal{F}_m$-measurable, we can apply Exercise 1.5(a) to get $E[X_m \mid \mathcal{F}_m] = X_m$, and hence

$$E[X_n \mid \mathcal{F}_m] = X_m,$$

as required.

(2) $\Leftrightarrow$ (3): Using linearity of conditional expectations (see Exercise 1.5(d)), we write

$$E[X_{k+1} - X_k \mid \mathcal{F}_k] = E[X_{k+1} \mid \mathcal{F}_k] - E[X_k \mid \mathcal{F}_k].$$

Since X is adapted, then X_k is $\mathcal{F}_k$-measurable, and hence (by Exercise 1.5(a)) $E[X_k \mid \mathcal{F}_k] = X_k$. Thus, the above equality becomes

$$E[X_{k+1} - X_k \mid \mathcal{F}_k] = E[X_{k+1} \mid \mathcal{F}_k] - X_k,$$

from which it immediately follows that (2) $\Leftrightarrow$ (3).

Exercise 3.4 *(Discrete stochastic integral)* The aim of this exercise is to prove Theorem 3.1.

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_k)_{k=0,\dots,T}$. Let $X = (X_k)_{k=0,\dots,T}$ and $\vartheta = (\vartheta_k)_{k=1,\dots,T}$ be $\mathbb{R}^d$-valued processes. Define the *stochastic integral process* $\vartheta \bullet X = (\vartheta \bullet X_k)_{k=0,\dots,T}$ by

$$\vartheta \bullet X_k := \sum_{j=1}^{k} \vartheta_j^{\mathrm{tr}} \Delta X_j = \sum_{j=1}^{k} \vartheta_j^{\mathrm{tr}} (X_j - X_{j-1}).$$

(a) Suppose X is a martingale, and ϑ is predictable and bounded. Show that $\vartheta \bullet X$ is a martingale.

What is $E[\vartheta \bullet X_T]$?

(b) Suppose X is a local martingale, and ϑ is predictable. Show that $\vartheta \bullet X$ is a local martingale.

Solution 3.4

(a) For $k \geqslant 1$, we have by assumption that X_j and ϑ_j are $\mathcal{F}_k$-measurable for all $j \leqslant k$. Hence $\vartheta \bullet X_k$ is $\mathcal{F}_k$-measurable, so that $\vartheta \bullet X$ is adapted. Next, since X is a martingale, it is integrable, and hence so is $X_j - X_{j-1}$ for all j. Since ϑ is bounded, it follows that $\vartheta \bullet X$ is integrable. It remains to show that

$$E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] = 0, \qquad \forall k = 0, \dots, T-1.$$

To this end, fix $k \in \{0, \dots, T-1\}$, and note that

$$\vartheta \bullet X_{k+1} - \vartheta \bullet X_k = \vartheta_{k+1}^{\mathrm{tr}} (X_{k+1} - X_k).$$

Since ϑ is predictable, then ϑ_{k+1} is $\mathcal{F}_k$-measurable. Because also φ_{k+1} is bounded and $X_{k+1} - X_k$ is integrable, we can "take out what is known" and write

$$\begin{aligned} E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] &= E[\vartheta_{k+1}^{\mathrm{tr}}(X_{k+1} - X_k) \mid \mathcal{F}_k] \\ &= \vartheta_{k+1}^{\mathrm{tr}} E[X_{k+1} - X_k \mid \mathcal{F}_k] \\ &= 0. \end{aligned}$$

(The property of conditional expectations that we used is as follows: if X and Y are integrable random variables such that XY is also integrable, and if Y is $\mathcal{G}$-measurable, then $E[XY \mid \mathcal{G}] = YE[X \mid \mathcal{G}]$. For a proof, come to the exercise class.) Thus, $\vartheta \bullet X$ is a martingale, as required. Finally, note that for any martingale Y, we have

$$E[Y_T] = E\Big[E[Y_T \mid \mathcal{F}_0]\Big] = E[Y_0],$$

where the first equality follows from Sheet 1, Exercise 5(b). In our setting, we have $\vartheta \bullet X_0 = 0$, and thus $E[\vartheta \bullet X_T] = 0$.

(b) By the same reasoning as in (a), $\vartheta \bullet X$ is an adapted process. Since it is null at zero, it remains to find a localising sequence. To this end, define the random variables

$$\sigma_n := \inf\{k : |\vartheta_{k+1}| > n\} \wedge T, \qquad n \in \mathbb{N}.$$

Since ϑ is predictable, then ϑ_{k+1} is $\mathcal{F}_k$-measurable, and thus by Exercise 3.1(f), $\inf\{k : |\vartheta_{k+1}| > n\}$ is a stopping time. By Exercise 3.1(a), we can take the minimum with T and still have a stopping time, because any nonnegative integer is a stopping time. Thus, σ_n is a stopping time. Now, since X is a local martingale, it has a localising sequence, say (τ_n). Define $\rho_n := \tau_n \wedge \sigma_n$, which is a stopping time by Exercise 3.1(a). We claim that (ρ_n) is a localising sequence for $\vartheta \bullet X$. To this end, we note that for each $\omega \in \Omega$, there is some $N \in \mathbb{N}$ such that $|\vartheta_{k+1}| \leqslant N$ for all k. It follows that $\sigma_n \uparrow T$ as $n \to \infty$. By definition, $\tau_n \uparrow T$ as $n \to \infty$, and thus also $\rho_n \uparrow T$ as $n \to \infty$. It remains to show that for each $n \in \mathbb{N}$, $\vartheta \bullet X^{\rho_n}$ is a martingale. We write

$$\vartheta \bullet X_k^{\rho_n} = \sum_{j=1}^{k \wedge \rho_n} \vartheta_j^{\mathrm{tr}}(X_j - X_{j-1}).$$

For $j > \rho_n$, we have $X_{j\wedge\rho_n} - X_{(j-1)\wedge\rho_n} = 0$, and thus we have

$$\begin{aligned} \vartheta \bullet X_k^{\rho_n} &= \sum_{j=1}^{k} \vartheta_{j\wedge\rho_n}^{\mathrm{tr}}(X_{j\wedge\rho_n} - X_{(j-1)\wedge\rho_n}) \\ &= \sum_{j=1}^{k} \mathbb{1}_{[0,\rho_n]}(j)\vartheta_j^{\mathrm{tr}} \Delta X_j^{\rho_n}, \end{aligned}$$

where $\mathbb{1}_{[0,\rho_n]} = \mathbb{1}_{[0,\rho_n]}(j)$ is a (random) function on $\{1,\dots,T\}$. Since (τ_n) is a localising sequence for X, then X^{τ_n} is a martingale. Because $\rho_n = \tau_n \wedge \sigma_n$, we have $X^{\rho_n} = (X^{\tau_n})^{\sigma_n}$, and thus X^{ρ_n} is a martingale (since a stopped martingale is a martingale). Moreover, by construction of ρ_n, we have that $|\mathbb{1}_{[0,\rho_n]}(k)\vartheta_k| \leqslant n$, so that $\mathbb{1}_{[0,\rho_n]}\vartheta = (\mathbb{1}_{[0,\rho_n]}(k)\vartheta_k)_{k=1,\dots,T}$ is a bounded, predictable process. We can thus apply part (a) to get that $\vartheta \bullet X^{\rho_n}$ is a martingale. This completes the proof.