Mathematical Foundations for Finance

Exercise Sheet 4

Please hand in your solutions by 12:00 on Wednesday, October 19 via the course homepage.

Exercise 4.1 *(Submartingales)* Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$.

(a) Let X be a martingale. Show that for any bounded and convex function $f \colon \mathbb{R} \to \mathbb{R}$, the process $f(X) = \big(f(X_k)\big)_{k \in \mathbb{N}_0}$ is a submartingale.

Could we replace "f is bounded" with a more general condition?

Hint: You may use that finite-valued convex functions are continuous.

(b) Let X be a submartingale, and let $\vartheta = (\vartheta_k)_{k \in \mathbb{N}}$ be a bounded, nonnegative and predictable process. Show that the stochastic integral process $\vartheta \bullet X$, defined by

$$\vartheta \bullet X_k = \sum_{j=1}^{k} \vartheta_j \Delta X_j = \sum_{j=1}^{k} \vartheta_j (X_j - X_{j-1}),$$

is a submartingale.

Conclude that $E[\vartheta \bullet X_k] \geqslant 0$ for all $k \in \mathbb{N}_0$.

(c) Let X be a submartingale and let τ be a stopping time. Show that the stopped process $X^\tau = (X_k^\tau)_{k \in \mathbb{N}_0}$ defined by $X_k^\tau = X_{k \wedge \tau}$ is a submartingale.

Exercise 4.2 *(Partition of sample space)* Let $\mathcal{P} = \{P_j : j \in J\}$ be a partition of a set Ω (i.e. a collection of disjoint nonempty sets with union Ω). The index set J here can be arbitrary. Show that the family

$$\mathcal{U}(\mathcal{P}) := \Big\{ \bigcup_{i \in I} P_i : I \subseteq J \Big\}.$$

consisting of all possible unions of sets P_j is a σ-field on Ω.

Note: When Ω is countable, the converse is also true; any σ-field on Ω is of the form $\mathcal{U}(\mathcal{P})$ for some partition $\mathcal{P}$ of Ω, where the set J is at most countable.

Exercise 4.3 *(Multinomial model)* Let $m \in \mathbb{N}$, and define the sample space Ω by

$$\Omega := \{1, \dots, m\}^T = \Big\{ \omega = (x_1, \dots, x_T) : x_k \in \{1, \dots, m\} \Big\}.$$

Fix some constants $p_1, \dots, p_m > 0$ with $\sum_{i=1}^m p_i = 1$. Set $\mathcal{F} = 2^\Omega$, and define the probability measure P on $(\Omega, \mathcal{F})$ by

$$P[\{\omega\}] = P[\{(x_1, \dots, x_T)\}] = \prod_{i=1}^T p_{x_i}, \qquad \omega = (x_1, \dots, x_T) \in \Omega.$$

Finally, pick some distinct constants $y_1, \dots, y_m$, and define the random variables $Y_k : \Omega \to \mathbb{R}$ by

$$Y_k(x_1, \dots, x_T) = y_{x_k}, \qquad k = 1, \dots, T.$$

(a) For $k = 1, \dots, T$ and $j = 1, \dots, m$, find $P[Y_k = y_j]$.

(b) Show that the random variables $Y_1, \dots, Y_T$ are i.i.d.

(c) Let $(\mathcal{F}_k)_{k=0,\dots,T}$ be the filtration generated by the process $Y = (Y_k)_{k=1,\dots,T}$, so that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and for each $k = 1, \dots, T$,

$$\mathcal{F}_k = \sigma(Y_1, \dots, Y_k).$$

Using the notation from Exercise 4.2, find a partition $\mathcal{P}_k$ of Ω such that $\mathcal{F}_k = \mathcal{U}(\mathcal{P}_k)$.

What is $\mathcal{F}_T$?

Exercise 4.4 *(Arbitrage opportunity)* Fix $u > d > -1$ and a finite time horizon $T \in \mathbb{N}$. Let $Y_1, \dots, Y_T$ be i.i.d. random variables with distribution given by

$$P[Y_k = 1 + u] = p, \qquad P[Y_k = 1 + d] = 1 - p,$$

where $p \in (0, 1)$ is fixed. Also, fix $r > -1$, and let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with the price processes of the assets in our market given by $\widetilde{S}^1_0 = 1$ and

$$\widetilde{S}^0_k = (1 + r)^k, \qquad k = 0, \dots, T,$$
$$\frac{\widetilde{S}^1_k}{\widetilde{S}^1_{k-1}} = Y_k, \qquad k = 1, \dots, T.$$

(a) By constructing an arbitrage opportunity, show that the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits arbitrage if $r \leqslant d$.

(b) Show that the same holds if $r \geqslant u$.