# Mathematical Foundations for Finance

## Exercise Sheet 4

*Please hand in your solutions by 12:00 on Wednesday, October 19 via the course homepage.*

**Exercise 4.1** *(Submartingales)* Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$.

(a) Let $X$ be a martingale. Show that for any bounded and convex function $f\colon \mathbb{R} \to \mathbb{R}$, the process $f(X) = (f(X_k))_{k\in\mathbb{N}_0}$ is a submartingale.

Could we replace "$f$ is bounded" with a more general condition?

*Hint: You may use that finite-valued convex functions are continuous.*

(b) Let $X$ be a submartingale, and let $\vartheta = (\vartheta_k)_{k\in\mathbb{N}}$ be a bounded, nonnegative and predictable process. Show that the stochastic integral process $\vartheta \bullet X$, defined by

$$\vartheta \bullet X_k = \sum_{j=1}^{k} \vartheta_j \Delta X_j = \sum_{j=1}^{k} \vartheta_j (X_j - X_{j-1}),$$

is a submartingale.

Conclude that $E[\vartheta \bullet X_k] \geqslant 0$ for all $k \in \mathbb{N}_0$.

(c) Let $X$ be a submartingale and let $\tau$ be a stopping time. Show that the stopped process $X^\tau = (X^\tau_k)_{k\in\mathbb{N}_0}$ defined by $X^\tau_k = X_{k\wedge\tau}$ is a submartingale.

### Solution 4.1

(a) The process $f(X)$ is integrable because $f$ is bounded. Since $X$ is adapted (because it is a martingale) and $f$ is continuous (since it is finite-valued and convex), it follows from Exercise 2.4 that $f(X)$ is adapted. It remains to establish the submartingale inequality. For $0 \leqslant m < n$, we write

$$E[f(X_n) \mid \mathcal{F}_m] \geqslant f(E[X_n \mid \mathcal{F}_m]) = f(X_m),$$

where the first step used the (conditional) Jensen's inequality, and the second step the martingale property. This concludes the proof.

A look at the proof shows that if we replace the condition "$f$ is bounded" by "$f(X)$ is integrable", the result still holds.

(b) Since $\vartheta$ is predictable and $X$ is adapted, then $\vartheta_j(X_j - X_{j-1})$ is $\mathcal{F}_j$-measurable for all $j \in \mathbb{N}$. It follows that $\vartheta \bullet X_k$ is $\mathcal{F}_k$-measurable, so that $\vartheta \bullet X$ is adapted. Also, since $\vartheta$ is bounded and $X$ is integrable, we have that $\vartheta \bullet X$ is integrable. It remains to establish the submartingale inequality. Note that (by the same reasoning as in Exercise 3.3) it suffices to show

$$E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] \geqslant 0, \qquad \forall k \in \mathbb{N}_0.$$

To this end, we write

$$\begin{aligned} E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] &= E[\vartheta_{k+1}(X_{k+1} - X_k) \mid \mathcal{F}_k] \\ &= \vartheta_k E[X_{k+1} - X_k \mid \mathcal{F}_k], \end{aligned}$$

where in the last step we used that $\vartheta_{k+1}$ is $\mathcal{F}_k$-measurable and bounded. Since $X$ is a submartingale, then $E[X_{k+1} - X_k \mid \mathcal{F}_k] \geqslant 0$. Since also $\vartheta_k$ is nonnegative by assumption, we have

$$E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] \geqslant 0,$$

as required.

Since $\vartheta \bullet X$ is a submartingale null at zero, we have for all $k \in \mathbb{N}_0$ that

$$E[\vartheta \bullet X_k] = E\Big[E[\vartheta \bullet X_k \mid \mathcal{F}_0]\Big] \geqslant E[\vartheta \bullet X_0] = 0.$$

(c) For $k \in \mathbb{N}_0$, we have

$$X_k^\tau = X_{k\wedge\tau} = X_0 + \sum_{j=1}^{k\wedge\tau}(X_j - X_{j-1}) = X_0 + \sum_{j=1}^{k} \mathbb{1}_{\{\tau \geqslant j\}}(X_j - X_{j-1}).$$

So if we set $\vartheta = (\vartheta_k)_{k\in\mathbb{N}}$ with $\vartheta_k := \mathbb{1}_{\{\tau\geqslant k\}}$, then

$$X_k^\tau = X_0 + \vartheta \bullet X_k, \qquad \forall k \in \mathbb{N}_0.$$

Since $\tau$ is a stopping time, then $\vartheta$ is a predictable process. Since $\vartheta$ is also bounded and nonnegative, and $X$ is a submartingale, we may apply part (b) to conclude that $\vartheta \bullet X$ is a submartingale. Also, note that because $X_0$ is $\mathcal{F}_0$-measurable and integrable, then the process $(X_0)_{k\in\mathbb{N}_0}$ is a submartingale (in fact a martingale). Since the sum of two submartingales is a submartingale, we can conclude that $X^\tau$ is a submartingale, as required.

**Exercise 4.2** *(Partition of sample space)* Let $\mathcal{P} = \{P_j : j \in J\}$ be a partition of a set $\Omega$ (i.e. a collection of disjoint nonempty sets with union $\Omega$). The index set $J$ here can be arbitrary. Show that the family

$$\mathcal{U}(\mathcal{P}) := \Big\{ \bigcup_{i\in I} P_i : I \subseteq J \Big\}.$$

consisting of all possible unions of sets $P_j$ is a $\sigma$-field on $\Omega$.

*Note: When $\Omega$ is countable, the converse is also true; any $\sigma$-field on $\Omega$ is of the form $\mathcal{U}(\mathcal{P})$ for some partition $\mathcal{P}$ of $\Omega$, where the set $J$ is at most countable.*

**Solution 4.2** We check the requirements for a $\sigma$-field:

- $\Omega = \bigcup_{j\in J} P_j \in \mathcal{U}(\mathcal{P})$.
- If $A = \bigcup_{j\in I} P_j \in \mathcal{U}(\mathcal{P})$, then $A^c = \bigcup_{j\in J\setminus I} P_j \in \mathcal{U}(\mathcal{P})$.
- If $A_n \in \mathcal{U}(\mathcal{P})$, where $A_n = \bigcup_{j\in J_n} P_j$, then

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{j\in J_n} P_j = \bigcup_{j\in \cup_{n=1}^{\infty} J_n} P_j \in \mathcal{U}(\mathcal{P}).$$

Thus, $\mathcal{U}(\mathcal{P})$ is a $\sigma$-field on $\Omega$.

**Exercise 4.3** *(Multinomial model)* Let $m \in \mathbb{N}$, and define the sample space $\Omega$ by

$$\Omega := \{1,\dots,m\}^T = \Big\{\omega = (x_1,\dots,x_T) : x_k \in \{1,\dots,m\}\Big\}.$$

Fix some constants $p_1,\dots,p_m > 0$ with $\sum_{i=1}^m p_i = 1$. Set $\mathcal{F} = 2^\Omega$, and define the probability measure $P$ on $(\Omega,\mathcal{F})$ by

$$P[\{\omega\}] = P[\{(x_1,\dots,x_T)\}] = \prod_{i=1}^{T} p_{x_i}, \qquad \omega = (x_1,\dots,x_T) \in \Omega.$$

Finally, pick some distinct constants $y_1,\dots,y_m$, and define the random variables $Y_k : \Omega \to \mathbb{R}$ by

$$Y_k(x_1,\dots,x_T) = y_{x_k}, \qquad k = 1,\dots,T.$$

(a) For $k = 1,\dots,T$ and $j = 1,\dots,m$, find $P[Y_k = y_j]$.

(b) Show that the random variables $Y_1,\dots,Y_T$ are i.i.d.

(c) Let $(\mathcal{F}_k)_{k=0,\dots,T}$ be the filtration generated by the process $Y = (Y_k)_{k=1,\dots,T}$, so that $\mathcal{F}_0 = \{\varnothing,\Omega\}$ and for each $k = 1,\dots,T$,

$$\mathcal{F}_k = \sigma(Y_1,\dots,Y_k).$$

Using the notation from Exercise 4.2, find a partition $\mathcal{P}_k$ of $\Omega$ such that $\mathcal{F}_k = \mathcal{U}(\mathcal{P}_k)$.

What is $\mathcal{F}_T$?

**Solution 4.3**

(a) We have

$$\begin{aligned}\{Y_k = y_j\} &= \{(x_1, \dots, x_T) \in \Omega : x_k = j\} \\ &= \{1, \dots, m\}^{k-1} \times \{j\} \times \{1, \dots, m\}^{T-k},\end{aligned}$$

and thus

$$\begin{aligned}P[Y_k = y_j] &= \sum_{(x_1,\dots,x_T)\in\{Y_k=y_j\}} \prod_{i=1}^{T} p_{x_i} \\ &= \sum_{x_1,\dots,x_{k-1},x_{k+1},\dots,x_T\in\{1,\dots,m\}} \prod_{i=1}^{k-1} p_{x_i} \times p_j \times \prod_{i=k+1}^{T} p_{x_i} \\ &= p_j \sum_{x_1,\dots,x_{k-1},x_{k+1},\dots,x_T\in\{1,\dots,m\}} \prod_{i=1}^{k-1} p_{x_i} \times \prod_{i=k+1}^{T} p_{x_i} \\ &= p_j \prod_{i=1}^{k-1} \sum_{x_i\in\{1,\dots,m\}} p_{x_i} \times \prod_{i=k+1}^{T} \sum_{x_i\in\{1,\dots,m\}} p_{x_i} \\ &= p_j.\end{aligned}$$

(b) It follows immediately from part (a) that the random variables $Y_1, \dots, Y_T$ are identically distributed. It remains to establish independence. That is, we need to show that for any $z_1, \dots, z_T \in \{y_1, \dots, y_m\}$,

$$P[Y_1 = z_1, \dots, Y_T = z_T] = \prod_{i=1}^{T} P[Y_i = z_i].$$

To this end, note that for each $i = 1, \dots, T$, there is a unique $j_i \in \{1, \dots, T\}$ such that $z_i = y_{j_i}$. This change of notation allows us to write

$$\{Y_1 = z_1, \dots, Y_T = z_T\} = \{Y_1 = y_{j_1}, \dots, Y_T = y_{j_T}\} = \{(j_1, \dots, j_T)\}.$$

Thus, we have (by the definition of $P$)

$$P[Y_1 = z_1, \dots, Y_T = z_T] = P[\{(j_1, \dots, j_T)\}] = \prod_{i=1}^{T} p_{j_i}.$$

Since $P[Y_i = z_i] = P[Y_i = y_{j_i}] = p_{j_i}$ by part (a), we therefore have

$$P[Y_1 = z_1, \dots, Y_T = z_T] = \prod_{i=1}^{T} P[Y_i = z_i],$$

as required.

(c) For each $k = 1, \dots, T$ and $x_1, \dots, x_k \in \{1, \dots, m\}$, define the set

$$\begin{aligned}A_{x_1,\dots,x_k} &= \{(x_1, \dots, x_k, x'_{k+1}, \dots, x'_T) : x'_{k+1}, \dots, x'_T \in \{1, \dots, m\}\} \\ &= \{(x_1, \dots, x_k)\} \times \{1, \dots, m\}^{T-k} \subseteq \Omega.\end{aligned}$$

For notational convenience, we also set $A_\varnothing := \Omega$. Next, for each $k = 0, \dots, T$, we define the family of sets

$$\mathcal{P}_k := \Big\{ A_{x_1,\dots,x_k} : x_1, \dots, x_k \in \{1, \dots, m\} \Big\}.$$

Note that $\mathcal{P}_0 = \{\Omega\}$, which is a partition of $\Omega$. Moreover, $\mathcal{U}(\mathcal{P}_0) = \{\varnothing, \Omega\} = \mathcal{F}_0$. Thus, for the remainder of the proof, we assume that $k > 0$.

It is easy to see that $\mathcal{P}_k$ is a partition of $\Omega$. First, each $A_{x_1,\dots,x_k}$ is clearly nonempty, and if $(x_1, \dots, x_k) \neq (x'_1, \dots, x'_k)$, then the elements of $A_{x_1,\dots,x_k}$ and $A_{x'_1,\dots,x'_k}$ must disagree somewhere on the first $k$ coordinates, so that $A_{x_1,\dots,x_k} \cap A_{x'_1,\dots,x'_k} = \varnothing$. Finally, if we take some element $(x_1, \dots, x_m) \in \Omega$, then $(x_1, \dots, x_m) \in A_{x_1,\dots,x_k}$, so that

$$\Omega = \bigcup_{A_{x_1,\dots,x_k} \in \mathcal{P}_k} A_{x_1,\dots,x_k}.$$

So $\mathcal{P}_k$ does indeed form a partition of $\Omega$. It remains to show that $\mathcal{F}_k = \mathcal{U}(\mathcal{P}_k)$. To this end, note that for each $A_{x_1,\dots,x_k} \in \mathcal{P}_k$, we have

$$A_{x_1,\dots,x_k} = \{Y_1 = y_{x_1}, \dots, Y_k = y_{x_k}\} \in \mathcal{F}_k,$$

so that $\mathcal{P}_k \subseteq \mathcal{F}_k$. Since $\Omega$ is finite, then all unions of sets in $\mathcal{P}$ are really *finite* unions, and thus $\mathcal{U}(\mathcal{P}_k) \subseteq \mathcal{F}_k$. Conversely, note that for each $\ell = 1, \dots, k$ and $j = 1, \dots, m$, we have

$$\begin{aligned}
\{Y_\ell = y_j\} &= \{1, \dots, m\}^{\ell-1} \times \{j\} \times \{1, \dots, m\}^{T-\ell} \\
&= \bigcup_{x_1,\dots,x_{\ell-1} \in \{1,\dots,m\}} A_{x_1,\dots,x_{\ell-1},j} \\
&= \bigcup_{x_1,\dots,x_{\ell-1},x_{\ell+1},\dots,x_k \in \{1,\dots,m\}} A_{x_1,\dots,x_{\ell-1},j,x_{\ell+1},\dots,x_k} \\
&\in \mathcal{U}(\mathcal{P}_k).
\end{aligned}$$

Since Exercise 4.2 gives that $\mathcal{U}(\mathcal{P}_k)$ is a $\sigma$-field, we can now say that $Y_\ell$ is $\mathcal{U}(\mathcal{P}_k)$-measurable for all $\ell = 1, \dots, k$. It then follows immediately from the definition of $\mathcal{F}_k$ that $\mathcal{F}_k \subseteq \mathcal{U}(\mathcal{P}_k)$, thus completing the proof.

We see that for each $(x_1, \dots, x_T) \in \Omega$,

$$A_{x_1,\dots,x_T} = \{(x_1, \dots, x_T)\},$$

and hence $\mathcal{F}_T = \mathcal{U}(\mathcal{P}_T) = 2^\Omega = \mathcal{F}$.

**Exercise 4.4** *(Arbitrage opportunity)* Fix $u > d > -1$ and a finite time horizon $T \in \mathbb{N}$. Let $Y_1, \dots, Y_T$ be i.i.d. random variables with distribution given by

$$P[Y_k = 1 + u] = p, \qquad P[Y_k = 1 + d] = 1 - p,$$

where $p \in (0,1)$ is fixed. Also, fix $r > -1$, and let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with the price processes of the assets in our market given by $\widetilde{S}^1_0 = 1$ and

$$\widetilde{S}^0_k = (1+r)^k, \qquad k = 0, \dots, T,$$
$$\frac{\widetilde{S}^1_k}{\widetilde{S}^1_{k-1}} = Y_k, \qquad k = 1, \dots, T.$$

(a) By constructing an arbitrage opportunity, show that the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits arbitrage if $r \leqslant d$.

(b) Show that the same holds if $r \geqslant u$.

**Solution 4.4**

(a) If $r \leq d$, the stock grows at least as fast as the bank account, and possibly faster since $u > d$. Formally, we have that for all $k = 1, \dots, T$,

$$Y_k \geqslant 1 + r, \qquad P[Y_k > 1 + r] > 0,$$

and therefore

$$S^1_k \geqslant S^1_{k-1}, \qquad P[S^1_k > S^1_{k-1}] > 0. \tag{1}$$

We therefore obtain an arbitrage opportunity as follows: at time 0, borrow money from the bank account to buy, say, one share of the stock, and hold it until time $T$. With probability 1, we will be able to completely repay the debt (by using the value of the stock), and with strictly positive probability, we will even have some money left over.

Formally, we are considering the self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$, where $V_0 = 0$ and $\vartheta = (\vartheta_k)_{k=0,\dots,T}$ is given by $\vartheta_0 = 0$ and $\vartheta_k = 1$ for $k = 1, \dots, T$. Since $\vartheta$ is deterministic, it is of course predictable, and thus $\varphi$ is a self-financing strategy by Proposition 2.3, and

$$V_k(\varphi) = V_0 + G_k(\vartheta) = \sum_{j=1}^{k} \vartheta_j \Delta S^1_j = \sum_{j=1}^{k} \Delta S^1_j = S^1_k - S^1_0 = S^1_k - 1.$$

Finally, (1) shows that

$$P[V_T(\varphi) \geqslant 0] = P[S^1_T \geqslant 1] = 1 \quad \text{and} \quad P[V_T(\varphi) > 0] > 0.$$

Hence, $\varphi$ is an arbitrage opportunity, as required.

(b) If $r \geqslant u$, then we consider the opposite strategy of part (a). At time zero, we sell short 1 share of the stock, and put the profits in the bank account. We hold our money in the bank until time $T$, and then buy the stock back to repay our debts. With probability 1, we will be able to repay our debt, in with strictly positive probability, we will have some money left over.

The explicit mathematical formulation of the arbitrage opportunity is very similar to part (a), and is therefore omitted.