ETH Zürich, Fall 2022
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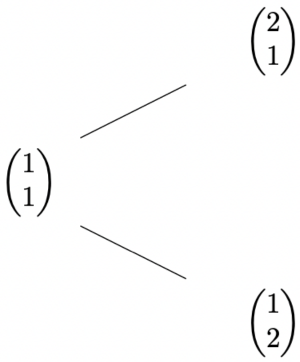
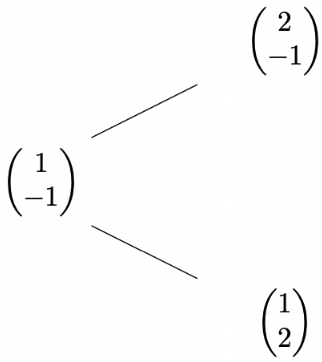
Mathematical Foundations for Finance

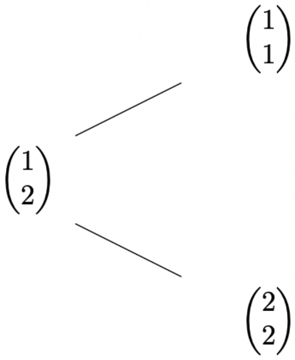
Exercise Sheet 5

Please hand in your solutions by 12:00 on Wednesday, October 26 via the course homepage.

Exercise 5.1 *(Types of arbitrage)* The following markets all have $T = 1$ and two risky assets (but no bank account) whose evolution is given in the respective picture (the transition probability corresponding to each branch is strictly positive). Check for each example whether or not it admits arbitrage of the first and of the second kind.

(a) 

(b) 

(c) 

Exercise 5.2 *(Admissibility and arbitrage)* The aim of this exercise is to show that we can drop the "admissibility" assumption from our definition of an arbitrage opportunity.

Let $\vartheta = (\vartheta_k)_{k=0,\ldots,T}$ be a predictable process with $\vartheta_0 = 0$, and let $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ be the corresponding self-financing strategy with initial capital 0.

(a) Show that if φ is not admissible, then there exists some $k \in \{0, \dots, T\}$ with $P[G_k(\vartheta) < 0] > 0$.

(b) Suppose that φ also satisfies $V_T(\varphi) \geqslant 0$ almost surely, and $P[V_T(\varphi) > 0] > 0$. Construct a modification φ' of φ so that φ' is an arbitrage opportunity. More precisely, find a predictable process ϑ' so that the corresponding self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$ is 0-admissible and satisfies $P[V_T(\varphi') > 0] > 0$.

Exercise 5.3 *(Stopped martingales)* Recall that for any time horizon, a stopped martingale is a martingale (by Corollary 3.2). In particular, if (for some $T \in \mathbb{N}$) $X = (X_k)_{k=0,\dots,T}$ is a martingale and $\tau : \Omega \to \{0, \dots, T\}$ is a stopping time, then $E[X_\tau] = E[X_{T\wedge\tau}] = E[X_T^\tau] = E[X_0^\tau] = E[X_0]$.

Show that this result does not hold in general for an infinite time horizon. That is, find an example of a martingale $X = (X_k)_{k\in\mathbb{N}_0}$ and a stopping time $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ with $P[\tau < \infty] = 1$, but where $E[X_\tau] \neq E[X_0]$.

Hint: For any (nondegenerate) random walk $Y = (Y_k)_{k\in\mathbb{N}_0}$, we have that with probability 1, $\liminf_{k\to\infty} Y_k = -\infty$ and $\limsup_{k\to\infty} Y_k = \infty$.

Exercise 5.4 *(Hitting times)* Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered measurable space, where $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. For each adapted process $X = (X_k)_{k\in\mathbb{N}_0}$ and each Borel set $B \in \mathcal{B}(\mathbb{R})$, define the function $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ by

$$\tau_{X,B} := \inf\{k \in \mathbb{N}_0 : X_k \in B\}.$$

This function $\tau_{X,B}$ is called the *(first) hitting time* of B.

(a) Show that $\tau_{X,B}$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a Borel set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$.

Note: This result shows that in discrete time, every stopping time is really the hitting time of some Borel set, for some adapted process.