ETH Zürich, Fall 2022 — Coordinator
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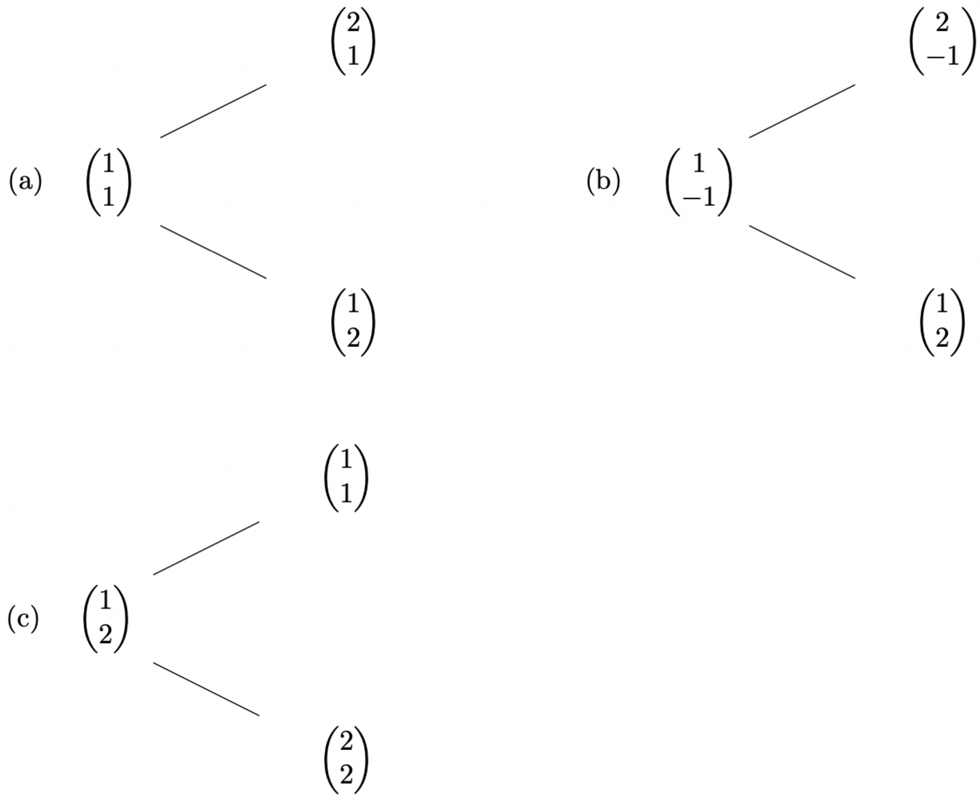
Mathematical Foundations for Finance

Exercise Sheet 5

Please hand in your solutions by 12:00 on Wednesday, October 26 via the course homepage.

Exercise 5.1 *(Types of arbitrage)* The following markets all have $T = 1$ and two risky assets (but no bank account) whose evolution is given in the respective picture (the transition probability corresponding to each branch is strictly positive). Check for each example whether or not it admits arbitrage of the first and of the second kind.

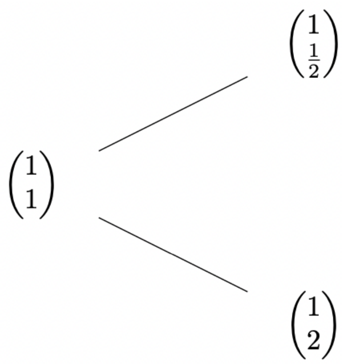


Solution 5.1

(a) Suppose that at time 0 we have a shares of stock 1 and b shares of stock 2, so that $V_0 = a + b$. Then at time 1, we have either that $V_1 = 2a + b$ or $V_1 = a + 2b$.

If there exists an arbitrage opportunity (first or second kind), then $a + b \leqslant 0$ and both $2a+b \geqslant 0$ and $a+2b \geqslant 0$. This implies that $a = (2a+b)-(a+b) \geqslant 0$ and $b = (a + 2b) - (a + b) \geqslant 0$. Together with $a + b \leqslant 0$, this yields $a = b = 0$, and so all of $-(a + b), 2a + b, a + 2b$ are zero (and hence none are strictly positive). Thus, there exist no arbitrage opportunities of either the first or the second kind.

Alternative solution via discounting: We can discount for instance by stock 1 to obtain the discounted prices:



This does not change the arbitrage properties of the model because the self-financing property of any strategy is invariant under discounting (see Exercise 2.2(c)). But now we can recognize the above as a binomial model with $u = 1$, $d = -\frac{1}{2}$, $r = 0$, which is arbitrage free because $u > r > d$.

(b) Suppose at time 0 we have $\frac{1}{2}$ shares of stock 1 and 1 share of stock 2, so that $V_0 = -\frac{1}{2} < 0$. Then at time 1, either $V_1 = \frac{1}{2} \cdot 2 - 1 = 0$ or $V_1 = \frac{1}{2} \cdot 1 + 2 = \frac{5}{2}$. This is an arbitrage opportunity of both the first kind and the second kind.

(c) Suppose at time 0 we have 1 share of stock 1 and $-\frac{3}{4}$ shares of stock 2, so that $V_0 = 1 - \frac{3}{4} \cdot 2 = -\frac{1}{2} < 0$. Then at time 1, either $V_1 = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$ or $V_1 = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$. This is an arbitrage opportunity of both the first kind and the second kind.

Exercise 5.2 *(Admissibility and arbitrage)* The aim of this exercise is to show that we can drop the "admissibility" assumption from our definition of an arbitrage opportunity.

Let $\vartheta = (\vartheta_k)_{k=0,\dots,T}$ be a predictable process with $\vartheta_0 = 0$, and let $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ be the corresponding self-financing strategy with initial capital 0.

(a) Show that if φ is not admissible, then there exists some $k \in \{0, \dots, T\}$ with $P[G_k(\vartheta) < 0] > 0$.

(b) Suppose that φ also satisfies $V_T(\varphi) \geqslant 0$ almost surely, and $P[V_T(\varphi) > 0] > 0$. Construct a modification φ' of φ so that φ' is an arbitrage opportunity. More precisely, find a predictable process ϑ' so that the corresponding self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$ is 0-admissible and satisfies $P[V_T(\varphi') > 0] > 0$.

Solution 5.2

(a) Suppose for contradiction that $P[G_k(\vartheta) \geqslant 0] = 1$ for all $k = 0, \dots, T$. Then since $V_0 = 0$ and φ is self-financing, we have

$$V_k(\varphi) = G_k(\vartheta) \geqslant 0, \qquad \forall k = 0, \dots, T,$$

so that φ is 0-admissible. This contradicts our assumption, and thus completes the proof.

(b) If φ is 0-admissible, we can take $\varphi' := \varphi$. So assume φ is not 0-admissible. We construct a new strategy $\phi' \mathrel{\widehat{=}} (0, \vartheta')$ as follows. Define

$$k_0 := \max\{k : P[G_k(\vartheta) < 0] > 0\}$$

to be the last time that $G(\vartheta)$ is strictly negative with some positive probability (which exists by assumption), and let

$$A := \{G_{k_0}(\vartheta) < 0\}$$

be the set on which this happens. Note that $k_0 \geqslant 1$ because $G_0(\vartheta) = 0$, and $k_0 \leqslant T-1$ because $G_T(\vartheta) = V_T(\varphi) \geqslant 0$ P-a.s. by assumption. We then define ϑ' so that the corresponding self-financing strategy φ' is to wait until time k_0, and then to follow φ on A, i.e.,

$$\vartheta'_k := \begin{cases} 0 & \text{if } k \leqslant k_0, \\ \vartheta_k \mathbb{1}_A & \text{if } k > k_0. \end{cases}$$

Since $A = \{G_{k_0}(\vartheta) < 0\} \in \mathcal{F}_{k_0}$ and ϑ is predictable, the product $\vartheta_k \mathbb{1}_A$ is $\mathcal{F}_{k-1}$-measurable for all $k > k_0$. Since $\vartheta'_k = 0$ for $k \leqslant k_0$, it follows that ϑ' is predictable, and thus indeed induces a self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$.

Next, we compute

$$V_k(\varphi') = G_k(\vartheta') = \sum_{j=1}^{k} (\vartheta'_j)^{\mathrm{tr}} \Delta S^1_j = \begin{cases} 0 & \text{if } k \leqslant k_0, \\ (G_k(\vartheta) - G_{k_0}(\vartheta)) \mathbb{1}_A & \text{if } k > k_0. \end{cases}$$

Since $G_{k_0}(\vartheta) < 0$ on A and $G_k(\vartheta) \geqslant 0$ almost surely for all $k > k_0$ by the maximality of k_0, it follows that $V_k(\varphi') \geqslant 0$ almost surely for all $k = 0, \dots, T$. Thus, φ is 0-admissible.

It now remains to check $P[V_T(\varphi') > 0] > 0$. To this end, we write

$$V_T(\varphi') = \Big(G_T(\vartheta) - G_{k_0}(\vartheta)\Big)\mathbb{1}_A = \Big(V_T(\varphi) - G_{k_0}(\vartheta)\Big)\mathbb{1}_A \geqslant -G_{k_0}(\vartheta)\mathbb{1}_A,$$

where we used that $V_T(\phi) \geqslant 0$ almost surely. Since $G_{k_0}(\vartheta) < 0$ on A, we have

$$P[V_T(\varphi') > 0] \geqslant P[-G_{k_0}(\vartheta)\mathbb{1}_A > 0] = P[A] > 0,$$

as required.

Exercise 5.3 *(Stopped martingales)* Recall that for any time horizon, a stopped martingale is a martingale (by Corollary 3.2). In particular, if (for some $T \in \mathbb{N}$) $X = (X_k)_{k=0,\dots,T}$ is a martingale and $\tau : \Omega \to \{0, \dots, T\}$ is a stopping time, then $E[X_\tau] = E[X_{T\wedge\tau}] = E[X_T^\tau] = E[X_0^\tau] = E[X_0]$.

Show that this result does not hold in general for an infinite time horizon. That is, find an example of a martingale $X = (X_k)_{k\in\mathbb{N}_0}$ and a stopping time $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ with $P[\tau < \infty] = 1$, but where $E[X_\tau] \neq E[X_0]$.

Hint: For any (nondegenerate) random walk $Y = (Y_k)_{k\in\mathbb{N}_0}$, we have that with probability 1, $\liminf_{k\to\infty} Y_k = -\infty$ and $\limsup_{k\to\infty} Y_k = \infty$.

Solution 5.3 Let $Z_1, Z_2, \dots$ be i.i.d. random variables with common distribution given by $P[Z_1 = -1] = P[Z_1 = 1] = \frac{1}{2}$. For each $k \in \mathbb{N}_0$, set

$$X_k := \sum_{j=1}^{k} Z_j.$$

The process $X = (X_k)_{k\in\mathbb{N}_0}$ is the *simple random walk* on $\mathbb{Z}$. Let $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ be the filtration generated by the Z_k, so that

$$\mathcal{F}_k := \sigma(Z_1, \dots, Z_k), \qquad \forall k \in \mathbb{N}_0.$$

We claim that X is an $\mathbb{F}$-martingale. By construction, X is $\mathbb{F}$-adapted, and it is integrable since X_k is the sum of k integrable random variables. Finally, we compute

$$E[X_{k+1} - X_k \mid \mathcal{F}_k] = E[Z_{k+1} \mid \mathcal{F}_k] = E[Z_{k+1}] = 0,$$

where the second equality uses that Z_{k+1} is independent of $Z_1, \dots, Z_k$. Thus, by Exercise 3.3, X is a martingale.

Next, define the map $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ by

$$\tau := \inf\{k \in \mathbb{N}_0 : X_k \geqslant 1\}.$$

By Exercise 3.1, τ is an $\mathbb{F}$-stopping time. Using the hint, we know that with probability 1, the simple random walk will "hit" 1 (or any integer) eventually, so that $P[\tau < \infty] = 1$.

Now, since $\tau < \infty$ almost surely, then $X_\tau \geqslant 1$ almost surely (by construction of τ), and hence

$$E[X_\tau] \geqslant 1 > 0 = E[X_0],$$

so that $E[X_\tau] \neq E[X_0]$.

Exercise 5.4 *(Hitting times)* Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered measurable space, where $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. For each adapted process $X = (X_k)_{k\in\mathbb{N}_0}$ and each Borel set $B \in \mathcal{B}(\mathbb{R})$, define the function $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ by

$$\tau_{X,B} := \inf\{k \in \mathbb{N}_0 : X_k \in B\}.$$

This function $\tau_{X,B}$ is called the *(first) hitting time* of B.

(a) Show that $\tau_{X,B}$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a Borel set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$.

Note: This result shows that in discrete time, every stopping time is really the hitting time of some Borel set, for some adapted process.

Solution 5.4

(a) The proof of (a) is identical to that of Exercise 3.1(f). We simply note that for each $k \in \mathbb{N}_0$,

$$\{\tau_{X,B} \leqslant k\} = \bigcup_{j=1}^{k} \{X_j \in B\},$$

and for each j, $\{X_j \in B\} \in \mathcal{F}_j \subseteq \mathcal{F}_k$, since X is adapted. It follows immediately that $\{\tau_{X,B} \leqslant k\} \in \mathcal{F}_k$, and thus $\tau_{X,B}$ is a stopping time, as required.

(b) Let τ be a stopping time. Define the process $X = (X_k)_{k\in\mathbb{N}_0}$ by $X_k = \mathbb{1}_{\{\tau\leqslant k\}}$. Since τ is a stopping time, then $\{\tau \leqslant k\} \in \mathcal{F}_k$, and hence X is adapted. Let $B = \{1\} \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}
\tau_{X,B} &= \inf\{k \in \mathbb{N}_0 : X_k \in B\} = \inf\{k \in \mathbb{N}_0 : \mathbb{1}_{\{\tau\leqslant k\}} = 1\} \\
&= \inf\{k \in \mathbb{N}_0 : \tau \leqslant k\} \\
&= \tau.
\end{aligned}$$

This completes the proof.