Mathematical Foundations for Finance

Exercise Sheet 6

Please hand in your solutions by 12:00 on Wednesday, November 2 via the course homepage.

Exercise 6.1 *(Radon–Nikodým derivative)* Consider a measurable space $(\Omega, \mathcal{F})$ equipped with two probability measures $Q \ll P$. The Radon–Nikodým theorem asserts that there is a unique (up to P-a.s. equality) random variable $\mathcal{D}$ such that $D \geqslant 0$ P-a.s. and

$$Q[A] = E_P[\mathcal{D}\mathbb{1}_A], \qquad \forall A \in \mathcal{F}. \tag{1}$$

The random variable $\mathcal{D}$ is called the *Radon–Nikodým* derivative of Q with respect to P, and is thus denoted by $\frac{\mathrm{d}Q}{\mathrm{d}P} := \mathcal{D}$. By standard measure-theoretic induction, one can show that (1) implies that for all Q-integrable or nonnegative random variables Y,

$$E_Q[Y] = E_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}Y\right]. \tag{2}$$

(a) Now equip the measurable space $(\Omega, \mathcal{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ and define for each $k \in \mathbb{N}_0$ the random variable

$$Z_k := E_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P} \,\middle|\, \mathcal{F}_k\right].$$

Note that by the definition of the Radon–Nikodým derivative, we have

$$E_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\right] = E_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\mathbb{1}_\Omega\right] = Q[\Omega] = 1 < \infty,$$

so that $\frac{\mathrm{d}Q}{\mathrm{d}P}$ is P-integrable (and thus Z_k is well-defined). Prove that

$$Z_k = \left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_k},$$

i.e. that for all $A \in \mathcal{F}_k$, $Q[A] = E_P[Z_k\mathbb{1}_A]$.

(b) Prove that $Z_k > 0$ Q-a.s. for all $k \in \mathbb{N}_0$.

(c) Fix $n \in \mathbb{N}_0$ and suppose U is a Q-integrable and $\mathcal{F}_n$-measurable random variable. Prove the Bayes formula

$$E_Q[U \mid \mathcal{F}_k] = \frac{1}{Z_k}E_P[Z_nU \mid \mathcal{F}_k] \quad Q\text{-a.s. for } 0 \leqslant k \leqslant n.$$

(d) Prove that an $\mathbb{F}$-adapted process $N = (N_k)_{k\in\mathbb{N}_0}$ is a Q-martingale if and only if the process $ZN = (Z_k N_k)_{k\in\mathbb{N}_0}$ is a P-martingale.

Solution 6.1

(a) For all $A \in \mathcal{F}_k$, we have

$$E_P[Z_k \mathbb{1}_A] = E_P\left[E_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P} \,\middle|\, \mathcal{F}_k\right]\mathbb{1}_A\right] = E_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\mathbb{1}_A\right] = Q[A],$$

as required.

(b) We have

$$Q[Z_k = 0] = E_P[Z_k \mathbb{1}_{\{Z_k=0\}}] = E_P[0] = 0,$$

and hence $Q[Z_k > 0] = 1 - Q[Z_k = 0] = 1$, so that $Z_k > 0$ Q-a.s., as required.

(c) Fix $k \in \{0, \dots, n\}$. Since U is Q-integrable and $\mathcal{F}_n$-measurable, the same measure-theoretic argument as for (1) $\Rightarrow$ (2) yields from part (a) that

$$E_P[Z_n|U|] = E_Q[|U|] < \infty,$$

so that $Z_n U$ is P-integrable. Thus $E_P[Z_n U \mid \mathcal{F}_k]$ is a well-defined, P-integrable and $\mathcal{F}_k$-measurable random variable. Now consider the random variable $\frac{1}{Z_k}|E_P[Z_n U \mid \mathcal{F}_k]|$. By part (b), $Z_k > 0$ Q-a.s., and so $\frac{1}{Z_k}$ is well-defined outside of a Q-null set. Also, recall by the definition of the conditional expectation that $E_P[Z_n U \mid \mathcal{F}_k]$ is unique up to a P-null set. Since $Q \ll P$, then also $E_P[Z_n U \mid \mathcal{F}_k]$ is unique up to a Q-null set. Thus, $\frac{1}{Z_k}|E_P[Z_n U \mid \mathcal{F}_k]|$ is unique and well-defined up to a Q-null set (and thus we can take its Q-expectation). Since it is also $\mathcal{F}_k$-measurable, we can use part (a) to conclude that

$$\begin{aligned} E_Q\left[\frac{1}{Z_k}\Big|E_P[Z_n U \mid \mathcal{F}_k]\Big|\right] &= E_P\Big[\Big|E_P[Z_n U \mid \mathcal{F}_k]\Big|\Big] \leqslant E_P\Big[E_P[Z_n|U| \mid \mathcal{F}_k]\Big] \\ &= E_P[Z_n|U|] < \infty, \end{aligned}$$

so that $\frac{1}{Z_k}E_P[Z_n U \mid \mathcal{F}_k]$ is Q-integrable.

Next, fix $A \in \mathcal{F}_k$. Since $\frac{1}{Z_k}E_P[Z_n U \mid \mathcal{F}_k]\mathbb{1}_A$ is $\mathcal{F}_k$-measurable and Q-integrable, part (a) and the definition of the conditional expectation yield

$$E_Q\left[\frac{1}{Z_k}E_P[Z_n U \mid \mathcal{F}_k]\mathbb{1}_A\right] = E_P\Big[E_P[Z_n U \mid \mathcal{F}_k]\mathbb{1}_A\Big] = E_P[Z_n U \mathbb{1}_A] = E_Q[U_n \mathbb{1}_A],$$

because $U\mathbb{1}_A$ is Q-integrable and $\mathcal{F}_n$-measurable. Hence, we have for all $A \in \mathcal{F}_k$ that

$$E_Q\left[\frac{1}{Z_k}E_P[Z_n U \mid \mathcal{F}_k]\mathbb{1}_A\right] = E_Q[U\mathbb{1}_A],$$

and thus the result follows by the definition of the conditional expectation.

(d) First, note that part (a) yields

$$E_Q[|N_n|] = E_P[Z_n|N_n|] = E_P|Z_n N_n|],$$

so that N is Q-integrable if and only if ZN is P-integrable. Next, Z is adapted by construction and N by assumption. So also ZN is adapted.

Suppose that N is a Q-martingale. For all $A \in \mathcal{F}_k$, we have

$$E_P[Z_n N_n \mathbb{1}_A] = E_Q[N_n \mathbb{1}_A] = E_Q[N_k \mathbb{1}_A] = E_P[Z_k N_k \mathbb{1}_A],$$

where the first and third equalities use that $N_n \mathbb{1}_A$ and $N_k \mathbb{1}_A$ are $\mathcal{F}_n$- and $\mathcal{F}_k$-measurable, respectively, and are both Q-integrable. The second equality uses that N is a Q-martingale. It follows that ZN is a P-martingale.

For the converse, we use (c) and the P-martingale property of ZN to write $E_Q[N_n \mid \mathcal{F}_k] = \frac{1}{Z_k} E_P[Z_n N_n \mid \mathcal{F}_k] = \frac{1}{Z_k} Z_k N_k = N_k$ Q-a.s. This completes the proof.

Exercise 6.2 *(Hedging)* Consider an attainable payoff $H \in L^0_+(\mathcal{F}_T)$, meaning that there exists some admissible self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ with $V_T(\varphi) = H$ P-a.s. Prove (under no-arbitrage) that at each time $k = 0, \dots, T$, the value V_k^H of the European option with payoff H (at expiry T) is equal to the value of the replicating strategy, i.e.

$$V_k^H = V_k(\varphi) \quad P\text{-a.s. for all } k = 0, \dots, T.$$

Solution 6.2 We first claim that $V_k^H \geqslant V_k(\varphi)$ P-a.s. for all $k = 0, \dots, T$. To this end, suppose for contradiction that there exists some $k_0 \in \{0, \dots, T\}$ such that $P[V_{k_0}^H < V_{k_0}(\varphi)] > 0$. We set $A := \{V_{k_0}^H < V_{k_0}(\varphi)\}$. Note that since $V_T^H = V_T(\varphi)$ P-a.s., we have $k_0 \in \{0, \dots, T-1\}$. We construct a new self-financing strategy φ' as follows. Up to time k_0, we follow φ. After time k_0, we follow φ on A^c up to the expiry T, and on A we sell φ, buy the European option, and hold the option until the expiry T. More precisely, we define the process $\vartheta' = (\vartheta'_k)_{k=0,\dots,T}$ by $\vartheta'_0 = (0,0)$, and for all $k = 1, \dots, T$,

$$\vartheta'_k = \begin{cases} (\vartheta_k, 0) & \text{if } k \leqslant k_0, \\ (\vartheta_k, 0)\mathbb{1}_{A^c} + (0,1)\mathbb{1}_A & \text{if } k > k_0. \end{cases}$$

The first and second coordinates of ϑ'_k are the number of shares of the stock and the option at time k, respectively. Since $A \in \mathcal{F}_{k_0}$, ϑ is predictable, and constants are measurable with respect to any σ-field, it follows that ϑ' is predictable. Let $\varphi' \mathrel{\widehat{=}} (V_0, \vartheta')$ be the corresponding self-financing strategy with initial wealth V_0. Then we have

$$\begin{aligned}
V_T(\varphi') &= V_0 + G_T(\vartheta') = V_0 + \sum_{j=1}^{T} (\vartheta'_j)^{\mathrm{tr}} (\Delta S^1_j, \Delta V^H_j) \\
&= V_0 + \sum_{j=1}^{k_0} (\vartheta'_j)^{\mathrm{tr}} (\Delta S^1_j, \Delta V^H_j) + \sum_{j=k_0+1}^{T} (\vartheta'_j)^{\mathrm{tr}} (\Delta S^1_j, \Delta V^H_j) \\
&= V_0 + \sum_{j=1}^{k_0} \vartheta_j^{\mathrm{tr}} \Delta S^1_j + \mathbb{1}_{A^c} \sum_{j=k_0+1}^{T} \vartheta_j^{\mathrm{tr}} \Delta S^1_j + \mathbb{1}_A \sum_{j=k_0+1}^{T} \Delta V^H_j \\
&= V_0 + \mathbb{1}_{A^c} \sum_{j=1}^{T} \vartheta_j^{\mathrm{tr}} \Delta S^1_j + \mathbb{1}_A \sum_{j=1}^{k_0} \vartheta_j^{\mathrm{tr}} \Delta S^1_j + \mathbb{1}_A \sum_{j=k_0+1}^{T} \Delta V^H_j \\
&= \mathbb{1}_{A^c} V_T(\varphi) + \mathbb{1}_A V_{k_0}(\varphi) + \mathbb{1}_A \sum_{j=k_0+1}^{T} \Delta V^H_j \\
&= \mathbb{1}_{A^c} V_T(\varphi) + \mathbb{1}_A V_{k_0}(\varphi) + \mathbb{1}_A (V^H_T - V^H_{k_0}) \\
&= \mathbb{1}_{A^c} V_T(\varphi) + \mathbb{1}_A V_{k_0}(\varphi) + \mathbb{1}_A V_T(\varphi) - \mathbb{1}_A V^H_{k_0} \\
&= V_T(\varphi) + \mathbb{1}_A (V_{k_0}(\varphi) - V^H_{k_0}).
\end{aligned}$$

Since $V_{k_0}(\varphi) > V^H_{k_0}$ on A, it follows that

$$V_T(\varphi') \geqslant V_T(\varphi) \qquad \text{and} \qquad P[V_T(\varphi') > V_T(\varphi)] = P[A] > 0.$$

Since φ and φ' are both self-financing strategies with the same initial value V_0, we thus have an arbitrage opportunity. This is a contradiction, and so we must have that $V^H_k \geqslant V_k(\varphi)$ P-a.s. for all $k = 0, \dots, T$.

It remains to show that $V^H_k \leqslant V_k(\varphi)$ P-a.s. for all $k = 0, \dots, T$. To see this, we repeat the above argument with $A = \{V^H_{k_0} > V_{k_0}(\varphi)\}$, which yields a self-financing strategy φ' that has the same initial value V_0 as φ, but with $V_T(\varphi') \leqslant V_T(\varphi)$ P-a.s. and $P[V_T(\varphi') < V_T(\varphi)] > 0$. Therefore the strategy $\varphi - \varphi'$ yields an arbitrage opportunity, thus completing the proof.

Remark. The strategies which we construct in the above arguments are not necessarily admissible. But this is no problem because we work in finite discrete time.

Exercise 6.3 *(Put and call options)* Let (S^0, S^1) be the (discounted) binomial model with $T = 1$, $u > 0 > d > -1$, and $p \in (0, 1)$. Fix $K > 0$, and define the functions $h_C, h_P : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}
h_C(x) &:= (x - K)^+ := \max\{0, x - K\}, \\
h_P(x) &:= (K - x)^+ := \max\{0, K - x\}.
\end{aligned}$$

The European options with payoff functions h_C and h_P are called the *European call option* and the *European put option*, respectively.

(a) Construct a self-financing strategy $\varphi^C \mathrel{\widehat{=}} (V_0^C, \vartheta^C)$ such that

$$V_1(\varphi^C) = h_C(S_1^1).$$

Write down explicitly the values of V_0^C and ϑ_1^C.

(b) Construct a self-financing strategy $\varphi^P \mathrel{\widehat{=}} (V_0^P, \vartheta^P)$ such that

$$V_1(\varphi^P) = h_P(S_1^1).$$

Write down explicitly the values of V_0^P and ϑ_1^P.

(c) Prove the *put–call parity* relation

$$V_0^P - V_0^C = K - S_0^1.$$

(d) Compute $\lim_{K\to\infty} V_0^C$, $\lim_{K\downarrow 0} V_0^C$, $\lim_{K\to\infty} V_0^P$ and $\lim_{K\downarrow 0} V_0^P$.

Explain why these values are not surprising.

Solution 6.3

(a) Consider a self-financing strategy $\varphi^C \mathrel{\widehat{=}} (V_0^C, \vartheta^C)$. By definition,

$$V_1(\varphi^C) = V_0^C + \vartheta_1^C \Delta S_1^1.$$

Since (S^0, S^1) is the binomial model, we have that either $S_1^1 = (1+u)S_0^1$ or $S_1^1 = (1+d)S_0^1$. Also, since ϑ_1^C is $\mathcal{F}_0$-measurable, it is a constant (i.e. non-random). Thus, φ satisfies $V_1(\varphi^C) = h_C(S_1^1)$ if and only if

$$\begin{aligned} V_0^C + \vartheta_1^C u S_0^1 &= h_C\Big((1+u)S_0^1\Big), \\ V_0^C + \vartheta_1^C d S_0^1 &= h_C\Big((1+d)S_0^1\Big). \end{aligned}$$

Subtracting the two equalities and rearranging gives

$$\vartheta_1^C = \frac{h_C((1+u)S_0^1) - h_C((1+d)S_0^1)}{(u-d)S_0^1}.$$

It remains to find V_0^C, which we can do by substituting the value of ϑ_1^C into either of the two previous equalities (we choose the first one) to get

$$\begin{aligned} V_0^C &= h_C\Big((1+u)S_0^1\Big) - \vartheta_1^C u S_0^1 \\ &= h_C\Big((1+u)S_0^1\Big) - \frac{h_C((1+u)S_0^1) - h_C((1+d)S_0^1)}{(u-d)S_0^1} u S_0^1 \\ &= \frac{u}{u-d} h_C\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_C\Big((1+u)S_0^1\Big). \end{aligned}$$

Note. Since $\frac{u}{u-d} + \frac{-d}{u-d} = 1$ *and* $\frac{u}{u-d} \in (0,1)$*, we can also write* $V_0^C = E^*[h_C(S_1^1)]$*, where* E^* *denotes the expectation under the "risk-neutral" probability measure* P^* *given by*

$$P^*[S_1^1 = (1+d)S_0^1] = \frac{u}{u-d}, \qquad P^*[S_1^1 = (1+u)S_0^1] = 1 - \frac{u}{u-d} = \frac{-d}{u-d}.$$

(b) The same reasoning as in part (a) yields

$$\begin{aligned} \vartheta_1^P &= \frac{h_P((1+u)S_0^1) - h_P((1+d)S_0^1)}{u-d}, \\ V_0^P &= \frac{u}{u-d} h_P\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_P\Big((1+u)S_0^1\Big). \end{aligned}$$

Note. For the same risk-neutral probability measure P^* *as in part (a), we can write*

$$V_0^P = E^*[h_P(S_1^1)].$$

(c) First we compute, for $x \in \mathbb{R}$,

$$h_P(x) - h_C(x) = \max\{0, K - x\} - \max\{0, x - K\} = K - x.$$

Using this together with parts (a) and (b) yields

$$\begin{aligned} V_0^P - V_0^C &= \frac{u}{u-d} h_P\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_P\Big((1+u)S_0^1\Big) \\ &\quad - \frac{u}{u-d} h_C\Big((1+d)S_0^1\Big) - \frac{-d}{u-d} h_C\Big((1+u)S_0^1\Big) \\ &= \frac{u}{u-d}\Big(K - (1+d)S_0^1\Big) + \frac{-d}{u-d}\Big(K - (1+u)S_0^1\Big) \\ &= K - S_0^1, \end{aligned}$$

as required.

Alternatively, we could use the expectation under the risk-neutral measure to get

$$V_0^P - V_0^C = E^*[h_P(S_1^1) - h_C(S_1^1)] = E^*[K - S_1^1] = K - E^*[S_1^1].$$

We then compute

$$\begin{aligned} E^*[S_1^1] &= (1+d)S_0^1 P^*[S_1^1 = (1+d)S_0^1] + (1+u)S_0^1 P^*[S_1^1 = (1+u)S_0^1] \\ &= (1+d)S_0^1 \frac{u}{u-d} + (1+u)S_0^1 \frac{-d}{u-d} \\ &= S_0^1, \end{aligned}$$

and hence

$$V_0^P - V_0^C = K - S_0^1,$$

as required.

A third way to establish put-call parity is by using no-arbitrage.

Consider the following self-financing strategy. At time 0, buy one put option, sell one call option, and buy one share of the stock. Then hold until time 1. We have

$$V_1(\varphi) = h_P(S_1^1) - h_C(S_1^1) + S_1^1 = K - S_1^1 + S_1^1 = K.$$

By no-arbitrage, we must also have $V_0(\varphi) = K$, i.e.

$$V_0^P - V_0^C + S_0^1 = K,$$

as required.

(d) For each $x > 0$, we have

$$\lim_{K\to\infty} h_C(x) = 0, \quad \lim_{K\downarrow 0} h_C(x) = x, \quad \lim_{K\to\infty} h_P(x) = \infty, \quad \lim_{K\downarrow 0} h_P(x) = 0.$$

Using the formula for V_0^C from part (a), we have

$$\lim_{K\to\infty} V_0^C = 0$$

and

$$\lim_{K\downarrow 0} V_0^C = \frac{u}{u-d}(1+d)S_0^1 + \frac{-d}{u-d}(1+u)S_0^1 = S_0^1.$$

By using put-call parity, we therefore have

$$\lim_{K\to\infty} V_0^P = \infty \qquad \text{and} \qquad \lim_{K\downarrow 0} V_0^P = 0.$$

(Alternatively, we could also have used the formula for V_0^P from part (b).)

The intuition for these limits is as follows. The option to buy the stock at expiry for a very high price is essentially worthless, which explains $\lim_{K\to\infty} V_0^C = 0$. Conversely, the price of the option to sell the stock at expiry for a very high price should also be very high, which explains $\lim_{K\to\infty} V_0^P = \infty$. Next, the price of the option to buy the stock at expiry for a very low price should be close to the price of the stock, which explains $\lim_{K\downarrow 0} V_0^C = S_0^1$. Conversely, the option to sell the stock at expiry for a very low price is essentially worthless, which explains $\lim_{K\downarrow 0} V_0^P = 0$.