Coordinator
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Mathematical Foundations for Finance

Exercise Sheet 8

Please hand in your solutions by 12:00 on Wednesday, November 16 via the course homepage.

Exercise 8.1 *(Doob decomposition)* Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$. Let $X = (X_k)_{k \in \mathbb{N}_0}$ be an adapted and integrable process.

(a) Prove that there exist a martingale $M = (M_k)_{k \in \mathbb{N}_0}$ and an integrable and predictable process $A = (A_k)_{k \in \mathbb{N}_0}$ that are both null at zero, and such that

$$X = X_0 + M + A.$$

(b) Prove that M and A are unique up to P-a.s. equality.

Exercise 8.2 *(Geometric Brownian motion)* Fix constants $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion. Define the process $S = (S_t)_{t \geqslant 0}$ by

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

The process $S = (S_t)_{t \geqslant 0}$ is called a *geometric Brownian motion.*

Find $\lim_{t \to \infty} S_t$ (if it exists) for all possible parameter constellations.

Hint: You can use the law of the iterated logarithm.

Exercise 8.3 *(Stopping theorem)* Let W be a Brownian motion. Is it true that for all stopping times τ, $E[W_\tau] = E[W_0]$? Why or why not?

Exercise 8.4 *(Variation and quadratic variation)* Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with finite variation. Let (π_n) be a sequence of partitions of $[0, \infty)$ with $|\pi_n| \to 0$ as $n \to \infty$. Show that for every $T \in [0, \infty)$,

$$\lim_{n \to \infty} \sum_{t_i \in \pi_n} |f(t_{i+1} \wedge T) - f(t_i \wedge T)|^2 = 0.$$

Exercise 8.5 *(Brownian motion)* Is the sum of two Brownian motions again a Brownian motion?