Mathematical Foundations for Finance

Exercise Sheet 8

Please hand in your solutions by 12:00 on Wednesday, November 16 via the course homepage.

Exercise 8.1 *(Doob decomposition)* Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. Let $X = (X_k)_{k\in\mathbb{N}_0}$ be an adapted and integrable process.

(a) Prove that there exist a martingale $M = (M_k)_{k\in\mathbb{N}_0}$ and an integrable and predictable process $A = (A_k)_{k\in\mathbb{N}_0}$ that are both null at zero, and such that

$$X = X_0 + M + A.$$

(b) Prove that M and A are unique up to P-a.s. equality.

Solution 8.1 To simplify notation, we omit "P-a.s." from all equalities below.

(a) For each $k \in \mathbb{N}_0$, take

$$M_k = \sum_{j=1}^{k}(X_j - E[X_j \mid \mathcal{F}_{j-1}]).$$

It is immediate that M is adapted, integrable, and null at zero. For each $k \in \mathbb{N}$, we have

$$\begin{aligned} E[M_k - M_{k-1} \mid \mathcal{F}_{k-1}] &= E\Big[X_k - E[X_k \mid \mathcal{F}_{k-1}] \,\Big|\, \mathcal{F}_{k-1}\Big] \\ &= E[X_k \mid \mathcal{F}_{k-1}] - E[X_k \mid \mathcal{F}_{k-1}] \\ &= 0. \end{aligned}$$

Hence, M is a martingale. Next, for each $k \in \mathbb{N}_0$, we set

$$\begin{aligned} A_k &= X_k - X_0 - M_k = X_k - X_0 - \sum_{j=1}^{k}(X_j - E[X_j \mid \mathcal{F}_{j-1}]) \\ &= \sum_{j=1}^{k}(E[X_j \mid \mathcal{F}_{j-1}] - X_{j-1}). \end{aligned}$$

Then A is predictable with $A_0 = 0$, and of course $X = X_0 + M + A$, as required.

(b) Suppose the processes $M^{(1)}, A^{(1)}$ and $M^{(2)}, A^{(2)}$ both satisfy the conditions of the problem. Subtracting the equalities

$$X - X_0 = M^{(1)} + A^{(1)},$$
$$X - X_0 = M^{(2)} + A^{(2)}$$

gives

$$M^{(1)} - M^{(2)} = A^{(2)} - A^{(1)}.$$

For notational convenience, we set $Y := M^{(1)} - M^{(2)} = A^{(2)} - A^{(1)}$. Since $Y = A^{(2)} - A^{(1)}$, then Y is predictable, and hence for all $k \in \mathbb{N}$,

$$Y_k = E[Y_k \mid \mathcal{F}_{k-1}].$$

But since the difference of two martingales is a martingale, Y is a martingale, and hence the above can be rewritten as

$$Y_k = Y_{k-1}, \qquad \forall k \in \mathbb{N}.$$

Since $Y_0 = 0$, this implies that $Y_k = 0$ for all $k \in \mathbb{N}_0$, and hence

$$M^{(1)} = M^{(2)} \qquad \text{and} \qquad A^{(1)} = A^{(2)}.$$

This completes the proof.

Exercise 8.2 *(Geometric Brownian motion)* Fix constants $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion. Define the process $S = (S_t)_{t\geqslant 0}$ by

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

The process $S = (S_t)_{t\geqslant 0}$ is called a *geometric Brownian motion.*

Find $\lim_{t\to\infty} S_t$ (if it exists) for all possible parameter constellations.

Hint: You can use the law of the iterated logarithm.

Solution 8.2 We can rewrite S_t as

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{2t\log\log t}\frac{W_t}{\sqrt{2t\log\log t}}\right) \\ &= S_0 \exp\left(\sqrt{2t\log\log t}\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{t}{\sqrt{2t\log\log t}} + \sigma\frac{W_t}{\sqrt{2t\log\log t}}\right)\right). \end{aligned}$$

Since

$$\lim_{t\to\infty}\sqrt{2t\log\log t} = \infty \qquad \text{and} \qquad \lim_{t\to\infty}\frac{t}{\sqrt{2t\log\log t}} = \infty,$$

and by the law of the iterated logarithm, it follows that:

when $\mu > \frac{\sigma^2}{2}$,

$$\lim_{t\to\infty} S_t = \infty;$$

when $\mu < \frac{\sigma^2}{2}$,

$$\lim_{t\to\infty} S_t = 0;$$

when $\mu = \frac{\sigma^2}{2}$,

$$\liminf_{t\to\infty} S_t = 0 \qquad \text{and} \qquad \limsup_{t\to\infty} S_t = \infty,$$

and hence $\lim_{t\to\infty} S_t$ does not exist.

Alternative solution: We can write

$$\log\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

By the law of the iterated logarithm, W_t grows slower than t as $t \to \infty$. So for $\mu > \frac{\sigma^2}{2}$, $\log(S_t/S_0) \to \infty$, hence $S_t \to \infty$, and for $\mu < \frac{\sigma^2}{2}$, $\log(S_t/S_0) \to -\infty$, hence $S_t \to 0$. For $\mu = \frac{\sigma^2}{2}$, $\log(S_t/S_0) = \sigma W_t$ has $\limsup_{t\to\infty} \sigma W_t = \infty$ and $\liminf_{t\to\infty} \sigma W_t = -\infty$, hence $\limsup_{t\to\infty} S_t = \infty$ and $\liminf_{t\to\infty} S_t = 0$, so that $\lim_{t\to\infty} S_t$ does not exist.

Exercise 8.3 *(Stopping theorem)* Let W be a Brownian motion. Is it true that for all stopping times τ, $E[W_\tau] = E[W_0]$? Why or why not?

Solution 8.3 No. Consider the stopping time

$$\tau := \inf\{t \geqslant 0 : W_t = 1\}.$$

We know from the law of the iterated logarithm that $\tau < \infty$ a.s., and thus $W_\tau = 1$ a.s. Hence,

$$E[W_\tau] = 1 \neq 0 = E[W_0].$$

Exercise 8.4 *(Variation and quadratic variation)* Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function with finite variation. Let (π_n) be a sequence of partitions of $[0,\infty)$ with $|\pi_n| \to 0$ as $n \to \infty$. Show that for every $T \in [0,\infty)$,

$$\lim_{n\to\infty} \sum_{t_i\in\pi_n} |f(t_{i+1} \wedge T) - f(t_i \wedge T)|^2 = 0.$$

Solution 8.4 Fix $n \in \mathbb{N}$. For each $t_i \in \pi_n$, we have $|t_{i+1} \wedge T - t_i \wedge T| \leqslant |\pi_n|$, and thus

$$\begin{aligned}\sum_{t_i \in \pi_n} |f(t_{i+1} \wedge T) - f(t_i \wedge T)|^2 &\leqslant \sup_{\substack{x,y \in [0,T] \\ |y-x| \leqslant |\pi_n|}} |f(y) - f(x)| \sum_{t_i \in \pi_n} |f(t_{i+1} \wedge T) - f(t_i \wedge T)| \\ &\leqslant \sup_{\substack{x,y \in [0,T] \\ |y-x| \leqslant |\pi_n|}} |f(y) - f(x)| V_T^1(f).\end{aligned}$$

Since f is continuous on the compact set $[0,T]$, it is uniformly continuous on $[0,T]$, and because also $|\pi_n| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \sup_{\substack{x,y \in [0,T] \\ |y-x| \leqslant |\pi_n|}} |f(y) - f(x)| = 0.$$

Since $V_T^1(f) < \infty$ by assumption, the conclusion follows.

Exercise 8.5 *(Brownian motion)* Is the sum of two Brownian motions again a Brownian motion?

Solution 8.5 No. Let W be a Brownian motion. Then $-W$ is also a Brownian motion, but $W - W = 0$ is not a Brownian motion.

Another example is given by $W + W = 2W$, because the distribution of $2W_t$ is normal with mean zero and variance $4t$ (and not variance t).