Mathematical Foundations for Finance

Exercise Sheet 9

Please hand in your solutions by 12:00 on Wednesday, November 23 via the course homepage.

Exercise 9.1 *(Square integrable martingales)* Let $M = (M_t)_{t \geqslant 0}$ be a square-integrable martingale with respect to P and $\mathbb{F}$, so that $E[M_t^2] < \infty$ for all $t \geqslant 0$.

(a) Show that for all $0 \leqslant s \leqslant t$,

$$E[(M_t - M_s)^2 \mid \mathcal{F}_s] = E[M_t^2 - M_s^2 \mid \mathcal{F}_s].$$

(b) Show that for all $0 \leqslant s \leqslant t \leqslant u \leqslant v$,

$$E[(M_t - M_s)(M_v - M_u)] = 0.$$

Exercise 9.2 *(Exit times of Brownian motion)* Let W be a Brownian motion with respect to P and $\mathbb{F}$, and fix finite constants $a < 0 < b$. Consider the function $\tau : \Omega \to [0, \infty]$ given by

$$\tau := \inf\{t \geqslant 0 : W_t \notin [a, b]\}.$$

The random variable τ is called the *exit time* of W from $[a, b]$.

(a) Show that τ is a stopping time.

(b) Prove that $E[W_\tau] = 0$.

(c) Find $P[W_\tau = a]$.

Exercise 9.3 *(Functions of Brownian motion)* Let W be a Brownian motion with respect to P and $\mathbb{F}$, and consider a measurable and bounded function $h : \mathbb{R} \to \mathbb{R}$. For each fixed $T > 0$, prove that there exists a function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that for all $0 \leqslant t \leqslant T$,

$$E[h(W_T) \mid \mathcal{F}_t] = g(t, W_t).$$

What is the explicit formula for $g(t, x)$?

What properties of Brownian motion did you use in your argument?

Hint: You may use that Lemma 2.2 in Chapter 8 of the lecture notes applies to measurable and bounded functions.

Exercise 9.4 *(Creating new Brownian motions)* Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion.

(a) Show that $-W$ is a Brownian motion.

(b) Fix $T \in (0,\infty)$. Show that $(W_{T+t} - W_T)_{t\geqslant 0}$ is a Brownian motion.

(c) Suppose that W' is a Brownian motion that is independent of W, and fix $\alpha \in (0,1)$. Show that $\alpha W + \sqrt{1-\alpha^2}W'$ is a Brownian motion.

(d) Give two counterexamples to show that the independence of W' and W in part (c) is necessary.