Non-Life Insurance: Mathematics and Statistics

Solution sheet 1

Solution 1.1 Discrete Distribution

(a) Note that N only takes values in $\mathbb{N}_{>0}$, and that $p \in (0,1)$. Hence, we calculate

$$\mathbb{P}[N \in \mathbb{R}] = \sum_{k=1}^{\infty} \mathbb{P}[N=k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=0}^{\infty} (1-p)^k = p \frac{1}{1-(1-p)} = 1,$$

from which we can conclude that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $n \in \mathbb{N}_{>0}$ we get

$$\mathbb{P}[N \geq n] = \sum_{k=n}^{\infty} \mathbb{P}[N=k] = \sum_{k=n}^{\infty} (1-p)^{k-1} p = (1-p)^{n-1} p \sum_{k=0}^{\infty} (1-p)^k = (1-p)^{n-1},$$

where we used that $p \sum_{k=0}^{\infty} (1-p)^k = 1$, as was shown in (a).

(c) The expectation of a discrete random variable that takes values in $\mathbb{N}_{>0}$ can be calculated (if it exists) as

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} k \cdot \mathbb{P}[N=k].$$

Thus, we get

$$\begin{aligned} \mathbb{E}[N] &= \sum_{k=1}^{\infty} k(1-p)^{k-1} p = \sum_{k=0}^{\infty} (k+1)(1-p)^k p = \sum_{k=0}^{\infty} k(1-p)^k p + \sum_{k=0}^{\infty} (1-p)^k p \\ &= (1-p)\mathbb{E}[N] + 1, \end{aligned}$$

where we again used that $p \sum_{k=0}^{\infty} (1-p)^k = 1$, as was shown in (a). If $\mathbb{E}[N]$ is finite, then we immediately conclude that

$$\mathbb{E}[N] = \frac{1}{p}.$$

In order to show that $\mathbb{E}[N]$ is indeed finite, we can use the ratio test on the series $\mathbb{E}[N] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$. Indeed,

$$\frac{(k+1)(1-p)^k p}{k(1-p)^{k-1} p} = (1-p)\frac{k+1}{k} \to (1-p) < 1$$

as $k \to \infty$. Therefore the series converges, and $\mathbb{E}[N] = \frac{1}{p}$.

(d) Let $r \in \mathbb{R}$. Then, we calculate

$$\begin{aligned} \mathbb{E}[\exp\{rN\}] &= \sum_{k=1}^{\infty} \exp\{rk\} \cdot \mathbb{P}[N=k] = \sum_{k=1}^{\infty} \exp\{rk\}(1-p)^{k-1} p \\ &= p \exp\{r\} \sum_{k=1}^{\infty} [(1-p)\exp\{r\}]^{k-1} = p \exp\{r\} \sum_{k=0}^{\infty} [(1-p)\exp\{r\}]^k. \end{aligned}$$

Since $(1-p)\exp\{r\}$ is strictly positive, the sum on the right hand side converges if and only if $(1-p)\exp\{r\} < 1$, which is equivalent to $r < -\log(1-p)$. Hence, $\mathbb{E}[\exp\{rN\}]$ exists if and only if $r < -\log(1-p)$, and in this case we have

$$M_N(r) \,=\, \mathbb{E}[\exp\{rN\}] \,=\, p\exp\{r\}\frac{1}{1-(1-p)\exp\{r\}} \,=\, \frac{p\exp\{r\}}{1-(1-p)\exp\{r\}}.$$

(e) For $r < -\log(1-p)$ we have

$$\begin{aligned}\frac{d}{dr}M_N(r) \,&=\, \frac{d}{dr}\frac{p\exp\{r\}}{1-(1-p)\exp\{r\}} \,=\, \frac{p\exp\{r\}[1-(1-p)\exp\{r\}]+p\exp\{r\}(1-p)\exp\{r\}}{[1-(1-p)\exp\{r\}]^2}\\ &=\, \frac{p\exp\{r\}}{[1-(1-p)\exp\{r\}]^2}.\end{aligned}$$

Hence, we get

$$\frac{d}{dr}M_N(r)\big|_{r=0} \,=\, \frac{p\exp\{0\}}{[1-(1-p)\exp\{0\}]^2} \,=\, \frac{p}{[1-(1-p)]^2} \,=\, \frac{p}{p^2} \,=\, \frac{1}{p}.$$

We observe that $\frac{d}{dr}M_N(r)\big|_{r=0} = \mathbb{E}[N]$, which holds in general for all random variables for which the moment generating function exists in an interval around 0.

Solution 1.2 Absolutely Continuous Distribution

(a) We calculate

$$\mathbb{P}[Y\in\mathbb{R}] \,=\, \int_{-\infty}^{\infty} f_Y(x)\,dx \,=\, \int_0^{\infty}\lambda\exp\{-\lambda x\}\,dx \,=\, [-\exp\{-\lambda x\}]_0^{\infty} \,=\, [-0-(-1)] \,=\, 1,$$

from which we can conclude that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $0 < y_1 < y_2$ we calculate

$$\begin{aligned}\mathbb{P}[y_1\le Y\le y_2] \,&=\, \int_{y_1}^{y_2} f_Y(x)\,dx \,=\, \int_{y_1}^{y_2}\lambda\exp\{-\lambda x\}\,dx \,=\, [-\exp\{-\lambda x\}]_{y_1}^{y_2}\\ &=\, \exp\{-\lambda y_1\}-\exp\{-\lambda y_2\}.\end{aligned}$$

(c) The expectation and the second moment of an absolutely continuous random variable can be calculated (if they exist) as

$$\mathbb{E}[Y] \,=\, \int_{-\infty}^{\infty} x f_Y(x)\,dx \qquad\text{and}\qquad \mathbb{E}[Y^2] \,=\, \int_{-\infty}^{\infty} x^2 f_Y(x)\,dx.$$

Thus, using partial integration, we get

$$\begin{aligned}\mathbb{E}[Y] \,&=\, \int_0^{\infty} x\lambda\exp\{-\lambda x\}\,dx \,=\, [-x\exp\{-\lambda x\}]_0^{\infty} + \int_0^{\infty}\exp\{-\lambda x\}\,dx\\ &=\, 0+\left[-\frac{1}{\lambda}\exp\{-\lambda x\}\right]_0^{\infty} \,=\, \frac{1}{\lambda}.\end{aligned}$$

The variance $\mathrm{Var}(Y)$ can be calculated as

$$\mathrm{Var}(Y) \,=\, \mathbb{E}[Y^2]-\mathbb{E}[Y]^2 \,=\, \mathbb{E}[Y^2]-\frac{1}{\lambda^2}.$$

For the second moment $\mathbb{E}[Y^2]$ we get, again using partial integration,

$$\begin{aligned}\mathbb{E}[Y^2] &= \int_0^\infty x^2 \lambda \exp\{-\lambda x\}\, dx = \left[-x^2 \exp\{-\lambda x\}\right]_0^\infty + \int_0^\infty 2x \exp\{-\lambda x\}\, dx \\ &= 0 + \frac{2}{\lambda}\mathbb{E}[Y] = \frac{2}{\lambda^2},\end{aligned}$$

from which we can conclude that

$$\mathrm{Var}(Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Note that for the exponential distribution both the expectation and the variance exist. The reason is that $\exp\{-\lambda x\}$ goes much faster to 0 than x or x^2 go to infinity, for all $\lambda > 0$.

(d) Let $r \in \mathbb{R}$. Then, we calculate

$$\mathbb{E}[\exp\{rY\}] = \int_0^\infty \exp\{rx\}\lambda \exp\{-\lambda x\}\, dx = \int_0^\infty \lambda \exp\{(r-\lambda)x\}\, dx.$$

The integral on the right hand side and therefore also $\mathbb{E}[\exp\{rY\}]$ exist if and only if $r < \lambda$. In this case we have

$$M_Y(r) = \mathbb{E}[\exp\{rY\}] = \frac{\lambda}{r-\lambda}\left[\exp\{(r-\lambda)x\}\right]_0^\infty = \frac{\lambda}{r-\lambda}(0-1) = \frac{\lambda}{\lambda - r},$$

and therefore

$$\log M_Y(r) = \log\left(\frac{\lambda}{\lambda - r}\right).$$

(e) For $r < \lambda$ we have

$$\frac{d^2}{dr^2}\log M_Y(r) = \frac{d^2}{dr^2}\log\left(\frac{\lambda}{\lambda-r}\right) = \frac{d^2}{dr^2}[\log(\lambda) - \log(\lambda - r)] = \frac{d}{dr}\frac{1}{\lambda - r} = \frac{1}{(\lambda-r)^2}.$$

Hence, we get

$$\frac{d^2}{dr^2}\log M_Y(r)|_{r=0} = \frac{1}{(\lambda-0)^2} = \frac{1}{\lambda^2}.$$

We observe that $\frac{d^2}{dr^2}\log M_Y(r)|_{r=0} = \mathrm{Var}(Y)$, which holds in general for all random variables for which the moment generating function exists in an interval around 0.

Solution 1.3 Gaussian Distribution

(a) Let $r \in \mathbb{R}$. Then, we calculate

$$\begin{aligned}M_X(r) &= \mathbb{E}\left[\exp\left\{rX\right\}\right] = \int_{-\infty}^\infty \exp\{rx\}\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\} dx \\ &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\frac{x^2 - 2(\mu + r\sigma^2)x + \mu^2}{\sigma^2}\right\} dx \\ &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\frac{x^2 - 2(\mu + r\sigma^2)x + \mu^2 + 2r\mu\sigma^2 + r^2\sigma^4 - 2r\mu\sigma^2 - r^2\sigma^4}{\sigma^2}\right\} dx \\ &= \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\frac{[x-(\mu + r\sigma^2)]^2}{\sigma^2}\right\} dx \\ &= \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\},\end{aligned}$$

where the last equality holds true since we integrate the density of a normal distribution with mean $\mu + r\sigma^2$ and variance σ^2.

(b) The moment generating function M_{a+bX} of $a+bX$ can be calculated as

$$M_{a+bX}(r) \,=\, \mathbb{E}\left[\exp\left\{r(a+bX)\right\}\right] \,=\, \exp\left\{ra\right\} \mathbb{E}\left[\exp\left\{rbX\right\}\right] \,=\, \exp\left\{ra\right\} M_X(rb),$$

for all $r \in \mathbb{R}$. Using the formula for the moment generating function of X given in part (a), we get

$$M_{a+bX}(r) \,=\, \exp\left\{ra\right\}\ \exp\left\{rb\mu + \frac{(rb)^2\sigma^2}{2}\right\} \,=\, \exp\left\{r(a+b\mu) + \frac{r^2b^2\sigma^2}{2}\right\},$$

which is equal to the moment generating function of a Gaussian random variable with expectation $a+b\mu$ and variance $b^2\sigma^2$. Since the moment generating function (if it exists in an interval around 0) uniquely determines the distribution, see Lemma 1.2 of the lecture notes (version of February 7, 2022), we conclude that

$$a+bX \,\sim\, \mathcal{N}(a+b\mu, b^2\sigma^2).$$

(c) Using the independence of $X_1, \ldots, X_n$, the moment generating function M_Y of $Y = \sum_{i=1}^n X_i$ can be calculated as

$$\begin{aligned} M_Y(r) \,&=\, \mathbb{E}\left[\exp\left\{rY\right\}\right] \,=\, \mathbb{E}\left[\exp\left\{r\sum_{i=1}^n X_i\right\}\right] \,=\, \prod_{i=1}^n \mathbb{E}\left[\exp\left\{rX_i\right\}\right] \,=\, \prod_{i=1}^n M_{X_i}(r) \\ &=\, \prod_{i=1}^n \exp\left\{r\mu_i + \frac{r^2\sigma_i^2}{2}\right\} \,=\, \exp\left\{r\sum_{i=1}^n \mu_i + \frac{r^2\sum_{i=1}^n\sigma_i^2}{2}\right\}, \end{aligned}$$

for all $r \in \mathbb{R}$. This is equal to the moment generating function of a Gaussian random variable with expectation $\sum_{i=1}^n \mu_i$ and variance $\sum_{i=1}^n \sigma_i^2$. We conclude that

$$\sum_{i=1}^n X_i \,\sim\, \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right).$$

Solution 1.4 χ^2-Distribution

(a) Let $r \in \mathbb{R}$. The moment generating function M_{X_k} of X_k can be calculated as

$$\begin{aligned} M_{X_k}(r) \,&=\, \mathbb{E}\left[\exp\{rX_k\}\right] \,=\, \int_0^\infty \exp\{rx\}\frac{1}{2^{k/2}\Gamma(k/2)}x^{k/2-1}\exp\{-x/2\}\,dx \\ &=\, \int_0^\infty \frac{1}{2^{k/2}\Gamma(k/2)}x^{k/2-1}\exp\{-x(1/2-r)\}\,dx. \end{aligned}$$

This integral (and consequently the moment generating function) exists if and only if $r < 1/2$. Let $r < 1/2$. Then, we use the substitution

$$u \,=\, x(1/2-r), \quad dx \,=\, \frac{1}{1/2-r}du.$$

We get

$$\begin{aligned} M_{X_k}(r) \,&=\, \int_0^\infty \frac{1}{2^{k/2}\Gamma(k/2)}u^{k/2-1}\left(\frac{1}{1/2-r}\right)^{k/2-1}\exp\{-u\}\frac{1}{1/2-r}\,du \\ &=\, \frac{1}{2^{k/2}}\frac{1}{(1/2-r)^{k/2}}\frac{1}{\Gamma(k/2)}\int_0^\infty u^{k/2-1}\exp\{-u\}\,du \\ &=\, \frac{1}{(1-2r)^{k/2}}, \end{aligned}$$

where in the last equality we used the definition of the gamma function

$$\Gamma(z) = \int_0^\infty u^{z-1} \exp\{-u\}\, du, \quad \text{for } z \in \mathbb{R}.$$

(b) For all $r < 1/2$ the moment generating function M_{Z^2} of Z^2 is given by

$$\begin{aligned} M_{Z^2}(r) &= \mathbb{E}\left[\exp\left\{rZ^2\right\}\right] = \int_{-\infty}^{\infty} \exp\left\{rx^2\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2(1-2r)}{2}\right\} dx \\ &= (1-2r)^{-1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}(1-2r)^{-1/2}} \exp\left\{-\frac{x^2}{2(1-2r)^{-1}}\right\} dx \\ &= \frac{1}{(1-2r)^{1/2}} \\ &= M_{X_1}(r), \end{aligned}$$

where the second to last equality holds true since we integrate the density of a normal distribution with mean 0 and variance $(1-2r)^{-1} > 0$. We conclude that $Z^2 \stackrel{(d)}{=} X_1$.

(c) Using that $Z_1, \ldots, Z_k$ are i.i.d., the moment generating function M_Y of $Y = \sum_{i=1}^k Z_i^2$ is given by

$$\begin{aligned} M_Y(r) &= \mathbb{E}\left[\exp\left\{rY\right\}\right] = \mathbb{E}\left[\exp\left\{r\sum_{i=1}^k Z_i^2\right\}\right] = \prod_{i=1}^k \mathbb{E}\left[\exp\left\{rZ_i^2\right\}\right] = \left(M_{Z_1^2}(r)\right)^k \\ &= \frac{1}{(1-2r)^{k/2}} = M_{X_k}(r), \end{aligned}$$

for all $r < 1/2$. We conclude that $\sum_{i=1}^k Z_i^2 \stackrel{(d)}{=} X_k$.