Non-Life Insurance: Mathematics and Statistics

Solution sheet 6

Solution 6.1 Log-Normal Distribution and Deductible

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, the moment generating function M_X of X is given by

$$M_X(r) = \mathbb{E}\left[\exp\{rX\}\right] = \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\},$$

for all $r \in \mathbb{R}$, see Exercise 1.3. Since Y_1 has a log-normal distribution with mean parameter μ and variance parameter σ^2, we have

$$Y_1 \overset{(d)}{=} \exp\{X\}.$$

Hence, the expectation, the variance and the coefficient of variation of Y_1 can be calculated as

$$\begin{aligned}
\mathbb{E}[Y_1] &= \mathbb{E}\left[\exp\{X\}\right] = \mathbb{E}\left[\exp\{1 \cdot X\}\right] = M_X(1) = \exp\left\{\mu + \frac{\sigma^2}{2}\right\},\\
\mathrm{Var}(Y_1) &= \mathbb{E}[Y_1^2] - \mathbb{E}[Y_1]^2 = \mathbb{E}\left[\exp\{2X\}\right] - M_X(1)^2 = M_X(2) - M_X(1)^2\\
&= \exp\left\{2\mu + \frac{4\sigma^2}{2}\right\} - \exp\left\{2\mu + 2\frac{\sigma^2}{2}\right\} = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left\{\sigma^2\right\} - 1\right) \quad \text{and}\\
\mathrm{Vco}(Y_1) &= \frac{\sqrt{\mathrm{Var}(Y_1)}}{\mathbb{E}[Y_1]} = \frac{\exp\left\{\mu + \sigma^2/2\right\}\sqrt{\exp\left\{\sigma^2\right\} - 1}}{\exp\left\{\mu + \sigma^2/2\right\}} = \sqrt{\exp\left\{\sigma^2\right\} - 1}.
\end{aligned}$$

(b) From part (a) we know that

$$\begin{aligned}
\sigma &= \sqrt{\log[\mathrm{Vco}(Y_1)^2 + 1]} \quad \text{and}\\
\mu &= \log \mathbb{E}[Y_1] - \frac{\sigma^2}{2}.
\end{aligned}$$

Since $\mathbb{E}[Y_1] = 3'000$ and $\mathrm{Vco}(Y_1) = 4$, we get

$$\begin{aligned}
\sigma &= \sqrt{\log(4^2 + 1)} \approx 1.68 \quad \text{and}\\
\mu &\approx \log 3'000 - \frac{(1.68)^2}{2} \approx 6.59.
\end{aligned}$$

(i) The claim frequency λ is given by $\lambda = \mathbb{E}[N]/v$. With the introduction of the deductible $d = 500$, the number of claims changes to

$$N^{\mathrm{new}} = \sum_{i=1}^{N} 1_{\{Y_i > d\}}.$$

Using the independence of N and $Y_1, Y_2, \ldots$, we get

$$\mathbb{E}[N^{\mathrm{new}}] = \mathbb{E}\left[\sum_{i=1}^{N} 1_{\{Y_i > d\}}\right] = \mathbb{E}[N]\mathbb{E}[1_{\{Y_1 > d\}}] = \mathbb{E}[N]\mathbb{P}[Y_1 > d].$$

Let Φ denote the distribution function of a standard Gaussian distribution. Since $\log Y_1$ has a Gaussian distribution with mean μ and variance σ^2, we have

$$\mathbb{P}[Y_1 > d] \,=\, 1 - \mathbb{P}[Y_1 \leq d] \,=\, 1 - \mathbb{P}\left[\frac{\log Y_1 - \mu}{\sigma} \leq \frac{\log d - \mu}{\sigma}\right] \,=\, 1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right).$$

Hence, the new claim frequency λ^{new} is given by

$$\lambda^{\text{new}} \,=\, \mathbb{E}[N^{\text{new}}]/v \,=\, \mathbb{E}[N]\mathbb{P}[Y_1 > d]/v \,=\, \lambda\mathbb{P}[Y_1 > d] \,=\, \lambda\left[1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)\right].$$

Inserting the values of d, μ and σ, we get

$$\lambda^{\text{new}} \,\approx\, \lambda\left[1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)\right] \,\approx\, 0.59 \cdot \lambda.$$

Note that the introduction of this deductible reduces the administrative burden a lot, because we expect that 41% of the claims disappear.

(ii) With the introduction of the deductible $d = 500$, the claim sizes change to

$$Y_i^{\text{new}} \,=\, Y_i - d \,|\, Y_i > d.$$

Thus, the new expected claim size is given by

$$\mathbb{E}[Y_1^{\text{new}}] \,=\, \mathbb{E}[Y_1 - d|Y_1 > d] \,=\, e(d),$$

where $e(d)$ is the mean excess function of Y_1 above d. According to page 75 of the lecture notes (version of February 7, 2022), $e(d)$ is given by

$$e(d) \,=\, \mathbb{E}[Y_1]\left[\frac{1 - \Phi\left(\frac{\log d - \mu - \sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)}\right] - d.$$

Inserting the values of d, μ, σ and $\mathbb{E}[Y_1]$, we get

$$\mathbb{E}[Y_1^{\text{new}}] \,\approx\, 3'000\left[\frac{1 - \Phi\left(\frac{\log 500 - 6.59 - 1.68^2}{1.68}\right)}{1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)}\right] - 500 \,\approx\, 4'456 \,\approx\, 1.49 \cdot \mathbb{E}[Y_1].$$

(iii) According to Proposition 2.2 of the lecture notes (version of February 7, 2022), the expected total claim amount $\mathbb{E}[S]$ is given by

$$\mathbb{E}[S] \,=\, \mathbb{E}[N]\mathbb{E}[Y_1].$$

With the introduction of the deductible $d = 500$, the total claim amount S changes to S^{new}, which can be written as

$$S^{\text{new}} \,=\, \sum_{i=1}^{N^{\text{new}}} Y_i^{\text{new}}.$$

Hence, the expected total claim amount changes to

$$\begin{aligned}\mathbb{E}[S^{\text{new}}] \,&=\, \mathbb{E}[N^{\text{new}}]\mathbb{E}[Y_1^{\text{new}}] \,=\, \mathbb{E}[N]\mathbb{P}[Y_1 > d]e(d) \,\approx\, 0.59 \cdot \mathbb{E}[N] \cdot 1.49 \cdot \mathbb{E}[Y_1] \\ &\approx\, 0.87 \cdot \mathbb{E}[S].\end{aligned}$$

In particular, the insurance company can grant a discount of roughly 13% on the pure risk premium. Note that also the administrative expenses on claims handling will reduce substantially because we only have 59% of the original claims, see the result in (i).

Solution 6.2 Akaike Information Criterion and Bayesian Information Criterion

(a) By definition, the MLEs $\left(\widehat{\gamma}^{\mathrm{MLE}}, \widehat{c}^{\mathrm{MLE}}\right)$ maximize the log-likelihood function $\ell_{\mathbf{Y}}$. In particular, we have

$$\ell_{\mathbf{Y}}\left(\widehat{\gamma}^{\mathrm{MLE}}, \widehat{c}^{\mathrm{MLE}}\right) \;\geq\; \ell_{\mathbf{Y}}\left(\gamma, c\right),$$

for all $(\gamma, c) \in \mathbb{R}_+ \times \mathbb{R}_+$.

If we write d^{MM} and d^{MLE} for the number of estimated parameters in the method of moments model and in the MLE model, respectively, we have $d^{\mathrm{MM}} = d^{\mathrm{MLE}} = 2$. The AIC value $\mathrm{AIC}^{\mathrm{MM}}$ of the method of moments model and the AIC value $\mathrm{AIC}^{\mathrm{MLE}}$ of the MLE model are then given by

$$\begin{aligned}
\mathrm{AIC}^{\mathrm{MM}} &= -2\ell_{\mathbf{Y}}\left(\widehat{\gamma}^{\mathrm{MM}}, \widehat{c}^{\mathrm{MM}}\right) + 2d^{\mathrm{MM}} \;=\; -2 \cdot 1\text{'}264.013 + 2 \cdot 2 \;=\; -2\text{'}524.026 \quad \text{and} \\
\mathrm{AIC}^{\mathrm{MLE}} &= -2\ell_{\mathbf{Y}}\left(\widehat{\gamma}^{\mathrm{MLE}}, \widehat{c}^{\mathrm{MLE}}\right) + 2d^{\mathrm{MLE}} \;=\; -2 \cdot 1\text{'}264.171 + 2 \cdot 2 \;=\; -2\text{'}524.342.
\end{aligned}$$

According to the AIC, the model with the smallest AIC value should be preferred. Since $\mathrm{AIC}^{\mathrm{MM}} > \mathrm{AIC}^{\mathrm{MLE}}$, we choose the MLE fit. Note that strictly speaking we should not use AIC to evaluate the MM estimated model since AIC only applies to MLE fitted models.

(b) If we write d^{gam} and d^{exp} for the number of estimated parameters in the gamma model and in the exponential model, respectively, we have $d^{\mathrm{gam}} = 2$ and $d^{\mathrm{exp}} = 1$. The AIC value $\mathrm{AIC}^{\mathrm{gam}}$ of the gamma model and the AIC value $\mathrm{AIC}^{\mathrm{exp}}$ of the exponential model are then given by

$$\begin{aligned}
\mathrm{AIC}^{\mathrm{gam}} &= -2\ell^{\mathrm{gam}}_{\mathbf{Y}}\left(\widehat{\gamma}^{\mathrm{MLE}}, \widehat{c}^{\mathrm{MLE}}\right) + 2d^{\mathrm{gam}} \;=\; -2 \cdot 1\text{'}264.171 + 2 \cdot 2 \;=\; -2\text{'}524.342 \quad \text{and} \\
\mathrm{AIC}^{\mathrm{exp}} &= -2\ell^{\mathrm{exp}}_{\mathbf{Y}}\left(\widehat{c}^{\mathrm{MLE}}\right) + 2d^{\mathrm{exp}} \;=\; -2 \cdot 1\text{'}264.169 + 2 \cdot 1 \;=\; -2\text{'}526.338.
\end{aligned}$$

Since $\mathrm{AIC}^{\mathrm{gam}} > \mathrm{AIC}^{\mathrm{exp}}$, we choose the exponential model.

The BIC value $\mathrm{BIC}^{\mathrm{gam}}$ of the gamma model and the BIC value $\mathrm{BIC}^{\mathrm{exp}}$ of the exponential model are given by

$$\mathrm{BIC}^{\mathrm{gam}} \;=\; -2\ell^{\mathrm{gam}}_{\mathbf{Y}}\left(\widehat{\gamma}^{\mathrm{MLE}}, \widehat{c}^{\mathrm{MLE}}\right) + d^{\mathrm{gam}} \cdot \log n \;=\; -2 \cdot 1\text{'}264.171 + 2 \cdot \log 1\text{'}000 \;\approx\; -2\text{'}514.53$$

and

$$\mathrm{BIC}^{\mathrm{exp}} \;=\; -2\ell^{\mathrm{exp}}_{\mathbf{Y}}\left(\widehat{c}^{\mathrm{MLE}}\right) + d^{\mathrm{exp}} \cdot \log n \;=\; -2 \cdot 1\text{'}264.169 + \log 1\text{'}000 \;\approx\; -2\text{'}521.43.$$

According to the BIC, the model with the smallest BIC value should be preferred. Since $\mathrm{BIC}^{\mathrm{gam}} > \mathrm{BIC}^{\mathrm{exp}}$, we choose the exponential model.

Note that the gamma model gives the better in-sample fit than the exponential model. But if we adjust this in-sample fit by the number of parameters used, we conclude that the exponential model probably has the better out-of-sample performance (better predictive power).

Solution 6.3 Goodness-of-Fit Test

(a) Let Y be a random variable following a Pareto distribution with threshold $\theta = 200$ and tail index $\alpha = 1.25$. Then, the distribution function G of Y is given by

$$G(x) \;=\; 1 - \left(\frac{x}{\theta}\right)^{-\alpha} \;=\; 1 - \left(\frac{x}{200}\right)^{-1.25},$$

for all $x \geq \theta$. For example for the interval I_2 we then have

$$\mathbb{P}[Y \in I_2] \;=\; \mathbb{P}[239 \leq Y < 301] \;=\; G(301) - G(239) \;=\; 0.2.$$

By analogous calculations for the other four intervals, we get

$$\mathbb{P}[Y \in I_1] = \mathbb{P}[Y \in I_2] = \mathbb{P}[Y \in I_3] = \mathbb{P}[Y \in I_4] = \mathbb{P}[Y \in I_5] \approx 0.2.$$

Let O_k denote the actual number of observations and E_k the expected number of observations in interval I_k, for all $k \in \{1, \ldots, 5\}$. The test statistic

$$X^2_{n,5} = \sum_{k=1}^{5} \frac{(O_k - E_k)^2}{E_k}$$

of the χ^2-goodness-of-fit test using $K = 5$ intervals and n observations converges to a χ^2-distribution with $K - 1 = 5 - 1 = 4$ degrees of freedom, as $n \to \infty$. As we have $n = 20$ observations in our data, we can calculate E_k as

$$E_k = 20 \cdot \mathbb{P}[Y \in I_k] = 20 \cdot 0.2 \approx 4,$$

for all $k = 1, \ldots, 5$. The values of the actual numbers of observations O_k and the expected numbers of observations E_k in the five intervals $k = 1, \ldots, 5$ as well as their squared differences $(O_k - E_k)^2$ are summarized in Table 1.

k	1	2	3	4	5
O_k	4	0	8	6	2
E_k	4	4	4	4	4
$(O_k - E_k)^2$	0	16	16	4	4

Table 1: Actual and expected numbers of observations with squared differences.

With the numbers in Table 1, the test statistic of the χ^2-goodness-of-fit test using 5 intervals in the case of our $n = 20$ observations is given by

$$X^2_{20,5} = \sum_{k=1}^{5} \frac{(O_k - E_k)^2}{E_k} = \frac{0}{4} + \frac{16}{4} + \frac{16}{4} + \frac{4}{4} + \frac{4}{4} = 10.$$

Let $\alpha = 5\%$. Then, the $(1 - \alpha)$-quantile of the χ^2-distribution with 4 degrees of freedom is given by approximately 9.49. Since this is smaller than $X^2_{20,5}$, we can reject the null hypothesis of having a Pareto distribution with threshold $\theta = 200$ and tail index $\alpha = 1.25$ as claim size distribution at significance level of 5%.

(b) We assume that we have n i.i.d. observations $Y_1, \ldots, Y_n$ from the null hypothesis distribution and that we work with $K = 2$ disjoint intervals I_1 and I_2. We define

$$p = \mathbb{P}[Y_1 \in I_1]$$

and

$$X_i = 1_{\{Y_i \in I_1\}},$$

for all $i = 1, \ldots, n$. This implies that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$. Thus, we have

$$\mu \overset{\text{def}}{=} \mathbb{E}[X_1] = p \qquad \text{and} \qquad \sigma \overset{\text{def}}{=} \sqrt{\text{Var}(X_1)} = \sqrt{p(1-p)}.$$

Moreover, we can write

$$O_1 = \sum_{i=1}^{n} X_i \qquad \text{and} \qquad O_2 = n - O_1 = n - \sum_{i=1}^{n} X_i$$

as well as

$$E_1 \;=\; \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] \;=\; np \qquad \text{and} \qquad E_2 \;=\; \mathbb{E}\left[n - \sum_{i=1}^{n} X_i\right] \;=\; n - np \;=\; n(1-p).$$

Therefore, we get

$$\begin{aligned} X^2_{n,2} \;&=\; \sum_{k=1}^{2} \frac{(O_k - E_k)^2}{E_k} \;=\; \frac{(O_1 - np)^2}{np} + \frac{[n - O_1 - n(1-p)]^2}{n(1-p)} \\ &=\; (O_1 - np)^2 \left[\frac{1}{np} + \frac{1}{n(1-p)}\right] \;=\; (O_1 - np)^2 \frac{1}{np(1-p)} \\ &=\; \left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}\right)^2. \end{aligned}$$

Let $Z \sim \mathcal{N}(0,1)$ and χ^2_1 follow a χ^2-square distribution with one degree of freedom. According to the central limit theorem, see equation (1.2) of the lecture notes (version of February 7, 2022), we have

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \;\Longrightarrow\; Z, \quad \text{as } n \to \infty.$$

As $Z^2 \overset{(d)}{=} \chi^2_1$, see Exercise 1.4, we can conclude that

$$X^2_{n,2} \;\Longrightarrow\; Z^2 \overset{(d)}{=} \chi^2_1, \quad \text{as } n \to \infty.$$

Solution 6.4 Kolmogorov-Smirnov Test

The distribution function G_0 of a Weibull distribution with shape parameter $\tau = \frac{1}{2}$ and scale parameter $c = 1$ is given by

$$G_0(y) \;=\; 1 - \exp\left\{-y^{1/2}\right\},$$

for all $y \geq 0$. Since G_0 is continuous, we are indeed allowed to apply a Kolmogorov-Smirnov test. If $x = (-\log u)^2$ for some $u \in (0,1)$, we have

$$G_0(x) \;=\; 1 - \exp\left\{-\left[(-\log u)^2\right]^{1/2}\right\} \;=\; 1 - \exp\{\log u\} \;=\; 1 - u.$$

Hence, if we evaluate G_0 at our data points $x_1, \ldots, x_5$, we get

$$G_0(x_1) \;=\; \frac{2}{40}, \quad G_0(x_2) \;=\; \frac{3}{40}, \quad G_0(x_3) \;=\; \frac{5}{40}, \quad G_0(x_4) \;=\; \frac{6}{40}, \quad G_0(x_5) \;=\; \frac{30}{40}.$$

We write $\widehat{G}_n$ for the empirical distribution function of a sample with n data points. The Kolmogorov-Smirnov test statistic D_n is then defined as

$$D_n \;=\; \sup_{y \in \mathbb{R}} \left|\widehat{G}_n(y) - G_0(y)\right|,$$

and $\sqrt{n}D_n$ converges to the Kolmogorov distribution K, as $n \to \infty$. The empirical distribution function $\widehat{G}_5$ of the sample $x_1, \ldots, x_5$ is given by

$$\widehat{G}_5(y) \;=\; \begin{cases} 0 & \text{if } y < x_1, \\ 1/5 & \text{if } x_1 \leq y < x_2, \\ 2/5 & \text{if } x_2 \leq y < x_3, \\ 3/5 & \text{if } x_3 \leq y < x_4, \\ 4/5 & \text{if } x_4 \leq y < x_5, \\ 1 & \text{if } y \geq x_5. \end{cases}$$

Since G_0 is continuous and strictly increasing with range $[0,1)$ and $\widehat{G}_5$ is piecewise constant and attains both the values 0 and 1, it is sufficient to consider the discontinuities of $\widehat{G}_5$ to determine the Kolmogorov-Smirnov test statistic D_5 for our $n = 5$ data points. We define

$$f(s-) \,=\, \lim_{r \nearrow s} f(r),$$

for all $s \in \mathbb{R}$, where the function f stands for G_0 and $\widehat{G}_5$. Since G_0 is continuous, we have $G_0(s-) = G_0(s)$ for all $s \in \mathbb{R}$. The values of G_0 and $\widehat{G}_5$ and their differences (in absolute value) are summarized in Table 2.

x_i, x_i-	x_1-	x_1	x_2-	x_2	x_3-	x_3	x_4-	x_4	x_5-	x_5
$\widehat{G}_5(\cdot)$	0	8/40	8/40	16/40	16/40	24/40	24/40	32/40	32/40	1
$G_0(\cdot)$	2/40	2/40	3/40	3/40	5/40	5/40	6/40	6/40	30/40	30/40
$\lvert\widehat{G}_5(\cdot) - G_0(\cdot)\rvert$	2/40	6/40	5/40	13/40	11/40	19/40	18/40	26/40	2/40	10/40

Table 2: Values of G_0 and $\widehat{G}_5$ and their differences (in absolute value).

From Table 2 we see for the Kolmogorov-Smirnov test statistic D_5 that

$$D_5 \,=\, \sup_{y \in \mathbb{R}} \left| \widehat{G}_5(y) - G_0(y) \right| \,=\, 26/40 \,=\, 0.65.$$

Let $q = 5\%$. By writing $K^{\leftarrow}(1-q)$ for the $(1-q)$-quantile of the Kolmogorov distribution, we have $K^{\leftarrow}(1-q) = 1.36$, see page 87 of the lecture notes (version of February 7, 2022). Since

$$\frac{K^{\leftarrow}(1-q)}{\sqrt{5}} \,\approx\, 0.61 \,<\, 0.65 \,=\, D_5,$$

we can reject the null hypothesis (at significance level of 5%) that the data $x_1, \ldots, x_5$ comes from a Weibull distribution with shape parameter $\tau = \frac{1}{2}$ and scale parameter $c = 1$.