O-minimality

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E. KOWALSKI

Organization:

(1) Home page

metaphor.ethz.ch/x/2022/hs/401-4037-72L

(2) Forum

forum.math.ethz.ch/c/autumn-2022/o-minimality-and-diophantine-applications/132

(3) Literature

L. van den Dries, "Tame topology and o-minimal structures"

A. Forey, lecture notes from ETH course, 2019

(4) Exercises: roughly every two weeks;

first exercise class 29.9
first exercise sheet today

Chapter I

What is o-minimality?

"O-minimality" refers to a type of geometry or topology in which the spaces and functions under consideration are restricted through a deceptively simple condition, which turns out to have surprisingly wide reach.

This notion is somewhat implicitly present in work of Tarski in the 1940's, concerning <u>semi-algebraic geometry</u>, but was only formalized in the 1980's by van den Dries, and developed as a field at the border of logic (model theory), geometry and topology until the early 2000's when J. Pila and U. Zannier realized that some of the ideas of o-minimality had very powerful arithmetic applications, which is extremely surprising.

I will explain (if not prove) some of these applications, but first we will need some time to build up the language involved in discussing o-minimality. To give a flavor of these applications, here is the simplest statement of that type:

<u>Theorem</u>. (Laurent)

Let $X \subset (\mathbb{C}^\times)^n$ be an irreducible algebraic variety ($=$ zero set of finitely many $f \in \mathbb{C}[(x_i, x_i^{-1})]$ which generate a prime ideal). Assume that

$$X_{tors} = \{ (x_i) \in X \mid \exists m \geq 1, \forall i, \ x_i^m = 1 \}$$

is <u>Zariski-dense</u> in $(\mathbb{C}^\times)^n$ (i.e., there is no $g \neq 0$ in $\mathbb{C}[(x_i, x_i^{-1})]$ vanishing on X_{tors}). Then there exist

(1) an $x_0 \in X_{tors}$

(2) a finite family $\underline{n} = (n_j)$ of elements of $\mathbb{Z}^n - \{0\}$

such that $X = x_0 H_{\underline{n}}$, where

$$H_{\underline{n}} = \left\{ x \in (\mathbb{C}^\times)^n \;\middle|\; \forall j, \quad \prod x_i^{n_{j,i}} = 1 \right\}$$

(an algebraic subgroup of $(\mathbb{C}^\times)^n$).

There are actually quite a few proof of Laurent's Theorem which are not so difficult, but many generalizations (Manin-Mumford, André-Oort, Pink-Zilber) are much more challenging (the last still open in general, the second only accessible to these o-minimal-based strategies).

<u>Remark</u>. Laurent's Theorem appears quite surprisingly in some recent works:

(1) "Space vectors forming rational angles", Kedlaya, Kolpakov, Poonen, Rubinstein

(classifies sets of non-zero $x \in \mathbb{R}^3$ with pairwise angles in $\mathbb{Q}\pi$)

(2) "Non-virtually abelian anisotropic linear groups are not boundedly generated",

Corvaja, Rapinchuk, Ren, Zannier

(if $\Gamma \subset GL_n(\mathbb{C})$ is of the form

$$\Gamma = \{ x_1^{n_1} \cdots x_m^{n_m} \mid n_i \in \mathbb{Z} \}$$

for $x_i \in \Gamma$ diagonalizable, then Γ has a finite-index solvable subgroup).