# Chapter II

# Model Theory

Although one can to some extent do without (as in the book of van den Dries), the "proper" framework to study o-minimality is through <u>model theory</u> and <u>"o-minimal structures"</u>. We present here a very quick introduction, with especially the so-called <u>compactness theorem</u> of first order logic.

## 1 _ <u>Language, structures</u>

The motivation can be seen as a super-generaliza-tion of algebraic geometry, where one defines a certain class of sets (zero sets of polyno-mials) by allowing other types of functions, or of relations (e.g. inequalities).

## Definitions.

(1) [Language / signature] A <u>language</u> $\mathcal{L}$ is a set or family of elements of the form:

$(\underline{f}, n)$ $\qquad \underline{f}$ "symbol", $n \geq 1$ $\qquad$ (function symbol)

$(\underline{r}, n)$ $\qquad \underline{r}$ "symbol", $n \geq 1$ $\qquad$ (relation symbol)

$\underline{c}$ $\qquad \underline{c}$ "symbol" $\qquad$ (constant symbol)

(2) [$\mathcal{L}$ - structure] Given a language $\mathcal{L}$, an <u>$\mathcal{L}$ - structure</u> is the data of:

- a set $M$ (the domain)
- for each function symbol $(\underline{f}, n)$ a map $\quad f_M : M^n \longrightarrow M$
- for each relation symbol $(\underline{r}, n)$ an $n$-ary relation, i.e. a subset $r_M \subset M^n$; one writes often $r(x_1, \dots, x_n)$ instead of $(x_1, \dots, x_n) \in r_M$.
- for each constant symbol $\underline{c}$, an element $c_M \in M$

Examples.

(1) $\mathcal{L} = \emptyset$ : the $\mathcal{L}$-structures are just sets.

(2) $\mathcal{L} = ((+,2), (-,1), (\cdot,2), \underline{0}, \underline{1})$

[usually abbreviated $(+,-,\cdot,0,1)$] is the language of rings. The $\mathcal{L}$-structures are sets $M$ with two named elements $0_M$ and $1_M$, and maps

$$\begin{cases} M \times M \xrightarrow{+} M \\ M \xrightarrow{-} M \\ M \times M \xrightarrow{\cdot} M \end{cases}.$$

and no further conditions [in particular, an $\mathcal{L}$-structure might have nothing to do with a ring!]

(3) $\mathcal{L} = \{ (\leq, 2) \}$ is the language of ordered sets: structures are sets given with a binary relation, usually denoted $x \leq y$ instead of $\leq(x,y)$ or $(x,y) \in \leq$.

Definitions. Let $\mathcal{L}$ be a language.

(1) [Morphism] If $M, N$ are $\mathcal{L}$-structures

Then a morphism from $M$ to $N$ is a map (of sets) $\varphi : M \longrightarrow N$ "respecting the language":

a) for any function symbol $(\underline{f}, n)$ and $x \in M^n$ we have $\varphi(f_M(x)) = f_N(\varphi(x))$

b) for any relation symbol $(\underline{r}, n)$ and $x \in M^n$ we have $(\underline{x} \in r_M \Rightarrow \varphi(x) \in r_N)$

c) for any constant symbol $\underline{c}$, we have $\varphi(c_M) = c_N$.

(2) [Embedding / substructure] An injective morphism $f$ is an embedding if b) is strengthened to $(\underline{x} \in r_M \Leftrightarrow \varphi(x) \in r_N)$. If $M \subset N$ and $M \hookrightarrow N$ is an embedding, then $M$ is a substructure of $N$. If $f$ is a bijective morphism, it is called an isomorphism.

Example. Let $\mathcal{L}$ be the language of rings. Any ring (with unit) $A$ gives a "natural" $\mathcal{L}$-structure by $+_A(x,y) = x+y$, $-_A(x) = -x$, $\cdot_A(x,y) = xy$, $0_A = 0$ and $1_A = 1$.

If $B$ is another ring, viewed similarly as an $\mathcal{L}$-structure, then any ring morphism $A \to B$ is a morphism of $\mathcal{L}$-structures.

(But if we defined $+_B(x,y) = xy$, $\cdot_B(x,y) = x+y$, then a morphism of $\mathcal{L}$-structures $A \xrightarrow{\varphi} B$ would be a strange thing: $\begin{cases} \varphi(x+y) = \varphi(x)\varphi(y) \ldots \\ \varphi(xy) = \varphi(x) + \varphi(y) \end{cases}$ )

## 2 - Terms, formulas, sentences

Now we will combine functions and relations with first order logic to create "statements" about the $\mathcal{L}$-structures.

Definitions - $\mathcal{L}$ a language ; $V$ = set of variables (disjoint from anything else!)

(1) [Term] An $\mathcal{L}$-term with variables in $V$ is a "syntactical" expression obtained by finitely many applications of the following rules:

(a) any variable $v \in V$, any $\underline{c}$, is a term

(b) if $(\underline{f}, n)$ is a function symbol and

$t_1, \dots, t_n$ are $\mathcal{L}$-terms then

$\underline{f}(t_1, \dots, t_n)$ is a term

[Ex. $\mathcal{L} = (+, -, \cdot, 0, 1)$ ; $+(+(x,1), \cdot(1, +(1,y)))$

is a term with variables $x, y$]

We often write $t(v_1, \dots, x_m)$ to say that

$t$ is a term with these variables.

(2) [Formula] A formula [with free variables $(v_i)$]

is a "syntactical" expression obtained by finitely many

applications of the following rules:

(a) for terms $t_1(\underline{v})$, $t_2(\underline{v})$, the expression

$t_1 = t_2$ "reserved" symbol! Should not be used in $\mathcal{L}$ or V...

is a formula [with free variables $(v_i)$].

(b) for terms $t_1(\underline{v}), \dots, t_m(\underline{v})$ and relation

symbol $(\underline{r}, m)$, the expression

$\underline{r}(t_1, \dots, t_m)$ is also one.

(c) [logical connectors] for formulas $\varphi_1(\underline{v}), \varphi_2(\underline{v})$, the expressions $\underset{\text{and}}{\varphi_1 \wedge \varphi_2}$, $\underset{\text{or}}{\varphi_1 \vee \varphi_2}$, $\underset{\text{implies}}{\varphi_1 \longrightarrow \varphi_2}$, $\underset{\text{not}}{\neg \varphi_1}$ are formulas [with free variables $\underline{v}$].

(d) [quantifiers] for a formula $\varphi(v_1, \underline{w})$, the expressions

$$\forall v_1, \varphi(v_1, \underline{w}) \qquad \text{for all}$$

$$\exists v_1, \varphi(v_1, \underline{w}) \qquad \text{there exists}$$

are formulas [with free variables $\underline{w}$].

$v_1$ is <u>not</u> free anymore

If a formula $\phi$ is constructed without using (d) it is called <u>quantifier-free</u>; if $\phi$ has no free variable, it is called a <u>sentence</u>.

<u>Examples.</u> (1) $\mathcal{L} = (<, 2)$ (ordered sets)

$$\forall x, (x < y) \vee (y < x)$$

is a formula with free variable $y$.

(2) $\mathcal{L} = ((\cdot, 2), e)$ (groups)

$$\forall x \forall y \forall z, \quad \cdot(\cdot(x,y), z) = \cdot(x, \cdot(y,z))$$

is a sentence.

## 3. Interpretation of formulas; definable sets

Let $\mathcal{L}$ be a language and $M$ an $\mathcal{L}$-structure. There is then an "obvious" way to "interpret" terms / formulas / sentences in $\mathcal{L}$ to define functions / sets:

(1) [Terms] if $t(v_1, \dots, v_m)$ is a term then its interpretation $t_M$ is a map

$$t_M : M^m \longrightarrow M$$

defined inductively from the definition of $t$ as follows:

(a) $(v_i)_M : \underline{x} \longmapsto x_i$

$c_M : \underline{x} \longmapsto c_M$

(b) for all function symbols $(\underline{f}, m)$ and terms $s_1(\underline{v}), \dots, s_m(\underline{v})$,

$$f(s_1, \dots, s_m)_M : \underline{x} \longmapsto f_M(s_{1,M}(\underline{x}), \dots, s_{m,M}(\underline{x}))$$

already interpreted

(2) For a formula $\varphi(v_1, \dots, v_m)$ [with these free variables], we have the <u>definable subset</u> $\varphi(M) \subset M^m$ defined inductively by

(a) If $\varphi =$ " $t_1 = t_2$ " then

$$\varphi(M) = \{ \underline{x} \in M^m \mid \underbrace{t_{1,M}}(\underline{x}) = \underbrace{t_{2,M}}(\underline{x}) \}$$

interpretations $M^m \to M$ of the terms $t_1, t_2$

(b) If $\varphi =$ " $\underline{r}(t_1, \dots, t_n)$ " for some relation symbol then

$$\varphi(M) = \{ \underline{x} \in M^m \mid r_M (t_{1,M}(\underline{x}), \dots, t_{n,M}(\underline{x})) \}$$

(c) <u>Connectors</u>:

$$(\varphi \wedge \psi)(M) = \varphi(M) \cap \psi(M)$$

$$(\varphi \vee \psi)(M) = \varphi(M) \cup \psi(M)$$

$$\neg \varphi(M) = M^m - \varphi(M)$$

$$(\varphi \to \psi)(M) = \varphi(M) \cup (M^m - \psi(M))$$

(d) <u>Quantifiers</u>: if $\varphi =$ " $\forall v_1 \psi(v_1, \underline{v})$ "

then $\varphi(M) = \{ \underline{x} \in M^m \mid \forall y \in M, (y, \underline{x}) \in \psi(M) \}$,

$$\psi(M) = \{ \underline{x} \in M^m \mid \exists y \in M, (y, \underline{x}) \in \varphi(M) \}.$$

<u>Notation</u>. If $\varphi$ is a <u>sentence</u> (no free variable), then $\varphi(M)$ is a subset of $M^0$, so is either empty or $M^0 = \{\text{single element}\}$. We say that "$\varphi$ holds in $M$" in the second case, denoted $M \models \varphi$. ("$M$ satisfies $\varphi$")

More generally, if $\varphi(x_1, \dots, x_n)$ is a formula and $a \in M^n$, then we write

$$M \models \varphi(a)$$

if and only if $a \in \varphi(M)$.

For a formula $\varphi(\underline{x}, \underline{y})$ with two sets of variables, and $b$ a $\underline{y}$-tuple in $M$, we write

$$\varphi(M, b) = \{ a \mid M \models \varphi(a, b) \}.$$

Sets of this form are called <u>definable with parameters</u>.

<u>Examples</u>. (1) $\mathcal{L} = (\leq, 2)$ (ordered sets)

$\varphi_1(y) =$ "$\forall x, \ x \leq y$"

$\varphi_2 = $ " $\exists y, \ \varphi_1(y)$ "

For an ordered set $M$, viewed as $\mathcal{L}$-structure in the natural way, $M \models \varphi_1(y)$ if and only if $y$ is the supremum of $M$, and $M \models \varphi_2$ if and only if $M$ has one.

For instance, $\mathcal{P}(X) \models \varphi_2$ for all sets $X$ ($\mathcal{P}(X)$ being the power set with inclusion).

[Note that if $\leq$ is interpreted differently as an order relation, the meaning of $\varphi_1$, $\varphi_2$ will be different!]

(2) $\mathcal{L} = (+, -, \cdot, 0, 1)$, the language of rings

Let $t(x_1, \dots, x_m)$ be an $\mathcal{L}$-term. From the definition, we see that there is a polynomial $f_t \in \mathbb{Z}[X_1, \dots, X_m]$ such that the interpretation of $t$ in any $\mathcal{L}$-structure $A$ which is indeed a unitary commutative ring, is the associated polynomial map $\quad (a_1, \dots, a_n) \longmapsto f_t(a_1, \dots, a_n)$.

It follows that <u>atomic</u> formulas (not involving either connectors or quantifiers) give definable sets

$$\varphi(A) = \{ a \in A^n \mid f_t(a) = 0 \}$$

i.e. the basic sets of algebraic geometry.

<u>Remark.</u> The rule $A \longmapsto \varphi(A)$ is reminiscent of the "functor of points" approach to algebraic geometry. One must however be careful because for a general formula $\varphi$ and morphism $A \to B$ (even embedding!) it is <u>not true</u> that we get a map $\varphi(A) \longrightarrow \varphi(B)$ !

For instance take $A = \mathbb{R}$, $B = \mathbb{C}$

$$\varphi(x) = \text{"}\forall y, \neg(x^2 = y)\text{"}$$

so that $\varphi(\mathbb{R}) = ]-\infty, 0[$ , $\varphi(\mathbb{C}) = \emptyset$

and there is no map $\varphi(\mathbb{R}) \longrightarrow \varphi(\mathbb{C})$.

In this example, still, the definable sets are well-understood by a theorem of Chevalley: there are finite boolean combination of the basic sets above ("constructible sets").

(3) Let $\mathcal{L}_{or} = (+, -, \cdot, 0, 1, \leq)$ be the language of ordered rings. Viewing $\mathbb{R}$ as an $\mathcal{L}_{or}$-structure in the obvious way, a fundamental theorem of Tarski–Seidenberg is

Th. Every definable set $\varphi(\mathbb{R})$ (with parameters) is a semi-algebraic set, i.e. a finite union of sets of the form

$$\{ x \in \mathbb{R}^n \mid f_1(x) = 0, f_2(x) > 0, \dots, f_n(x) > 0 \}$$

for some polynomials $f_i \in \mathbb{R}[X_1, \dots, X_n]$.

because we allow parameters

## 3 – O-minimal structure

We can define properly o-minimality.

Definition – [o-minimal structure]

Let $\mathcal{L}$ be a language containing a binary relation $\leq$. An $\mathcal{L}$-structure $M$ is <u>o-minimal</u> if

(1) the interpretation of $\leq$ in $M$ is a <u>total order</u>, <u>dense</u> $(\forall x < y, \exists z, x < z < y)$, <u>without endpoints</u> $((\neg \exists x, \forall y, y \leq x) \wedge (\neg \exists x, \forall y, x \leq y))$

(2) If $X \subset M$ is definable with parameters, then $X$ is a finite union of points and open intervals (i.e. $\{x \in M \mid a < x < b\}$)

<u>Examples</u> - (1) $\mathbb{Q}$, as ordered set, is an o-minimal structure.

(2) [Tarski-Seidenberg] $\mathbb{R}$, as $\mathcal{L}_{or}$-structure, is o-minimal. Indeed, the first condition is clear and if $X \subset \mathbb{R}$ is definable, then it is a finite union of sets of the

form

$$\{x \in \mathbb{R} \mid f_1(x) = 0,\ f_2(x) > 0, \dots, f_n(x) > 0\}$$

which are intersections of finite sets and sets $\{f(x) > 0\}$ for $f \in \mathbb{R}[X]$, and by basic calculus / topology, these are finite unions of intervals.

(3) One of the most important theorems in this area (including for arithmetic applications) is:

**Theorem.** (Wilkie, 1991)

Let $\mathcal{L} = (+, -, \cdot, 0, 1, \leq, \exp)$ and let $\mathbb{R}$ be the $\mathcal{L}$-structure where exp is interpreted as the usual exponential. Then $\mathbb{R}$ is an o-minimal $\mathcal{L}$-structure.

(We will state later an even more general fact).

(4) Here are some counterexamples...

For $\mathcal{L}_{\sin} = (+, -, \cdot, 0, 1, \leq, \sin)$, with $\sin$ interpreted as the sine function, the structure $\mathbb{R}$ is <u>not</u> o-minimal: the formula

$$\varphi(x) = \text{“} \sin(x) = 0 \text{”}$$

has $\varphi(\mathbb{R}) = 2\pi\mathbb{Z}$, which has $\infty$ly many connected components.

For $\mathcal{L} = (+, -, \cdot, 0, 1, \leq)$, the obvious $\mathcal{L}$-structure of $\mathbb{Q}$ is <u>not</u> o-minimal. One way to see this is to note that $\mathbb{Z}$ is definable in $\mathbb{Q}$; indeed, it is so in the language of rings:

<u>Th.</u> (J. Robinson). Let $\psi(a,b,m)$ be the formula "$\exists x \exists y \exists z \, (abm^2 + 2 = x^2 + ay^2 - bz^2)$" and let $\varphi(n)$ be the formula:

$$\forall a \forall b \left( \psi(a,b,0) \wedge (\forall m, \psi(a,b,m) \longrightarrow \psi(a,b,m+1)) \right) \longrightarrow \psi(a,b,n)$$

Then $\varphi(\mathbb{Q}) = \mathbb{Z}$.

A major related open problem is Hilbert's $10^{th}$ Problem over $\mathbb{Q}$: does there exist an algorithm that decides in finite time whether a polynomial $f \in \mathbb{Q}[X_1, \dots, X_n]$ has a zero in $\mathbb{Q}^n$? The link comes from the fact that the analogue question for $\mathbb{Z}$ has a negative answer (Davis, Putnam, Robinson, Matijasevic). This would give a negative answer for $\mathbb{Q}$ if $\mathbb{Z}$ was a diophantine set over $\mathbb{Q}$, meaning that there would exist $f \in \mathbb{Q}[X_., X_1, \dots, X_n]$ for some $n$ such that

$$\mathbb{Z} = \{ m \in \mathbb{Q} \mid \exists (x_1, \dots, x_n) \in \mathbb{Q}^n, f(m, \underline{x}) = 0 \}.$$

## 4. Theories, axiomatization

**Definition.** $\mathcal{L}$ language

(1) [Theory] An $\mathcal{L}$-theory is a set $T$ of $\mathcal{L}$-sentences; a model of $T$ is an $\mathcal{L}$-structure

$M$ such that $M \models \phi$ for all $\phi \in T$.

(2) [Satisfiability] An $\mathcal{L}$-theory $T$ is <u>satisfiable</u> if it has at least one model.

(3) [Theory of a structure] Let $M$ be an $\mathcal{L}$-structure. The <u>theory of $M$</u> is the $\mathcal{L}$-theory

$$Th(M) = \{ \mathcal{L}\text{-sentences } \phi \mid M \models \phi \}$$

(4) [Elementary classes ; axiomatization] The <u>elementary class</u> of $T$ is the class of all models of $T$ ; $T$ is an <u>axiomatization</u> of this class.

(5) [Elementary equivalence] $\mathcal{L}$-structures $M$, $N$ are elementarily equivalent $\iff Th(M) = Th(N)$.

<u>Examples</u>. (1) The theory of groups is the $\mathcal{L} = ((\cdot, 2), e)$ – theory given by the sentences

$\phi_1$: $\qquad \forall x \forall y \forall z,\ \cdot(x, \cdot(y,z)) = \cdot(\cdot(x,y), z)$

(associativity)

$\phi_2$: $\qquad \forall x,\ (\cdot(x,e) = x) \wedge (\cdot(e,x) = x)$

(e neutral)

$\phi_3$: $\forall x, \exists y \ (\cdot(x,y) = e) \wedge (\cdot(y,x) = e)$
(existence of inverse)

$T_{gp} = \{\phi_1, \phi_2, \phi_3\}$

An $\mathcal{L}_{gp}$-structure $M$ is a model of $T_{gp}$ if and only if $M$ is a group with product $\cdot_M$ and neutral element $e_M$.

Whether two groups are elementarily equivalent can be a difficult question! Only recently was it proved that free groups on $n \neq m$ generators are if $n, m \geqslant 2$ (it is clear that $F_1 \not\equiv F_m$ if $m \geqslant 2$, since $\phi_4$: $\forall x \forall y, \cdot(x,y) = \cdot(y,x)$ is a sentence satisfied by $F_1 \simeq \mathbb{Z}$ but not $F_n$ for $n \geqslant 2$). This was a problem of Tarski, solved independently by Sela and Kharlampovich-Myasnikov around 2006.

(2) In the same language we can axiomatize easily the class of groups where all elements

have order in some finite set $I$, by adding a sentence

$$\varphi_I : \quad \forall x \quad \underbrace{(x^{i_1}=e)}_{\text{short-hand for the obvious } \cdot(x,\cdot(x,\dots)) \text{ expression}} \vee (x^{i_2}=e) \vee \dots \vee (x^{i_n}=e)$$

However, it is not clear if/how to axiomatize the class of <u>torsion groups</u> in this language ($\exists n,\ x^n = e$ is <u>not</u> allowed because "$n$" does not belong to the language).

We will see later that, in fact, <u>there is no such axiomatization</u> (i.e, torsion groups are not elementary in this language).

(3) The classes of commutative rings or fields are elementary in the language of rings.

Furthermore, for $p=0$ or $p$ prime, the algebraically closed fields of characteristic $p$ models of a theory $ACF_p$. Indeed to the axioms of fields, one adds the sentence $\kappa_p: \forall x, \underbrace{x + \cdots + x}_{p \text{ times}} = 0$ (if $p$ is prime) and the sentences $\psi_d$ for all $d \geq 1$:

$$\psi_d : \quad \forall a_0 \forall a_1 \cdots \forall a_d \left( \overbrace{a_d \neq 0}^{\neg (a_d = 0)} \longrightarrow \exists x, \; a_0 + a_1 x + \cdots + a_d x^d = 0 \right)$$

One can show

<u>Theorem</u>. Let $K_1, K_2$ be algebraically closed fields. Then $K_1$ is elementarily equivalent to $K_2$ if and only if they have the same characteristic.

Note in particular how this shows that elementary equivalence may hold for structures of different cardinality.