Chapter V

O-minimal structures

1 - Basic definitions and notation

We consider a language $\mathcal{L}$ containing a relation $\leq$, interpreted as an ordering. An o-minimal $\mathcal{L}$-structure is in particular an $\mathcal{L}$-structure in which $\leq$ is interpreted as a dense linear order without endpoints.

We use this to define a natural topology on the underlying set M and its powers M^n, $n \geq 1$.

We will denote $\overline{M} = M \cup \{+\infty, -\infty\}$, with the "obvious" ordering coming from $\leq$.

Definition / Notation - (1) Intervals in M are given as for $\mathbb{R}$, and use the same notation, e.g.

$$]a, +\infty[\; = \{x \in M \mid x > a\}$$

(2) The (order-) topology on M is the topology

where intervals $]b,c[$ with $b<a<c$ form a basis of open neighborhoods of $a \in M$.

(3) The topology on M^m is the product topology.

(4) We simply say <u>o-interval</u> for non-empty open intervals, i.e. sets of the form [van den Dries just says "interval"]

$$\{x \in M \mid a < x < b\}, \quad a<b, \quad a,b \text{ in } \overline{M}.$$

(5) Let $m \geq 0$ and $X \subset M^m$ be definable. Let $n \geq 0$. A function $f: X \longrightarrow M^n$ is <u>definable</u> if its graph $\Gamma_f \subset M^{m+n}$ is definable. (The image of f is then also definable.) (This is a general definition)

Here are some first useful properties:

<u>Lemma</u>. (1) Let $X \subset M$ be definable. Then $\mathrm{Sup}(X)$ and $\mathrm{Inf}(X)$ exist in $\overline{M}$.

(2) Let $X \subset M$ be definable. The <u>boundary</u>

$$\partial X = \overline{X} - \mathring{X}$$

is finite; let $(a_i)_{0 \leq i \leq n}$ be the elements of $\overline{M}$ such that $\quad -\infty = a_0 < a_1 < \dots < a_n < a_{n+1} = \infty$

and $\partial X = \{ a_i \mid 1 \leq i \leq n \}$. Then for $0 \leq i \leq n$, the

interval $]a_i, a_{i+1}[$ is either contained in X or disjoint

from X.

(3) If $X \subset M^m$ is definable, so are $\overline{X}$ and $\mathring{X}$.

(4) If $X \subset M^m$ is open and definable, $f : X \longrightarrow M^n$ is

definable, then $\{ a \in M^m \mid f \text{ continuous at } a \}$ is

definable.

<u>Proof.</u> (1) is clear if we write

$$X = \{ a_1, \dots, a_m \} \cup \bigcup_{1 \leq j \leq n}]b_j, c_j[$$

for some $a_i \in M$, b_j, c_j in $\overline{M}$, by o-minimality. Then

$$\mathrm{Sup}(X) = \max(\{ a_i, c_j \}),$$

$$\mathrm{Inf}(X) = \min(\{ a_i, b_j \}).$$

(2) In $(*)$, we can assume $a_i \notin]b_j, c_j[$ and

$]b_j, c_j[\cap]b_k, c_k[= \emptyset$ if $k \neq \ell$. Then

$$\partial X = \{ a_i \} \cup \{ b_j \neq -\infty \} \cup \{ c_j \neq +\infty \}$$

and the property is clear.

(3) It suffices to define $\overline{X}$ (since $\overset{\circ}{X} = M^m - \left(\overline{M^m - X}\right)$) :

suppose $X = \varphi(M)$, then $\overline{X} = \overline{\varphi}(M)$, where

$$\overline{\varphi}(\underline{v}) : \quad \forall b_1 \dots \forall b_m \forall c_1 \dots \forall c_m \; \big(b_1 < v_1 < c_1 \wedge \dots \wedge b_m < v_m < c_m \big) \rightarrow \exists w_1 \dots \exists w_m , \; \varphi(\underline{w}) \wedge \big(b_1 < w_1 < c_1 \wedge \dots \wedge b_m < w_m < c_m \big)$$

(for all open neigh. of $\underline{v}$, there is a $\underline{w} \in \varphi(M)$ inside)

(4) Exercise...

$\square$

<u>Definition.</u> $X \subset M^m$ is <u>definably connected</u> if X is definable and is not the union of two disjoint non-empty open definable subsets of X.

<u>Remark.</u> This notion is needed because in certain o-minimal structures, M is <u>not</u> connected for the order topology, but is definably connected (if M is an ultrapower of $\mathbb{R}$, then $\{x \in M \mid \forall y \in \mathbb{R},\ 0 < y < x\}$, is open and closed for the order topology; here $\mathbb{R}$

is diagonally embedded in M).

Proposition – (1) The definably-connected subsets of M are the intervals of $\overline{M}$ (including $\emptyset$).

(2) If $a \leq b$ are in M and $f : [a,b] \longrightarrow M$ is continuous and definable, then $f([a,b])$ contains all x such that x lies between $f(a)$ and $f(b)$.

("Intermediate value theorem").

Proof – (1) This is easy because all definable sets are finite unions of points and o-intervals. Such a set is open if and only if it is a disjoint union of (finitely many) o-intervals.

First, if $X \subset M$ is not an interval, we find $a \in M$ such that $a \notin X$, $X \cap]a, +\infty[\neq \emptyset$, $X \cap]-\infty, a[\neq \emptyset$ and $X = (X \cap]-\infty, a[) \cup (X \cap]-\infty, a[)$ shows that X is not d-connected.

Conversely, let $X \subset M$ be an interval, and assume that $X = (X_1 \cap X) \cup (X_2 \cap X)$ where $X_i \subset M$ definable, open, not empty and $X_1 \cap X_2 \cap X = \emptyset$.

There is then an interval $]a,b[$, which is one of the components of X_1, such that $X \not\subset]a,b[$ but $X \cap]a,b[\neq \emptyset$. Suppose for instance that there is a $c \in X$ such that $c \geq b$. Note then also that $b \in X$ (otherwise $X \subset]b, +\infty[$, contradicting $X \cap]a,b[\neq \emptyset$). So $b \in X_2$; but since X_2 is open, we get $X_2 \cap]a,b[\neq \emptyset$; since also X is an interval and $X \cap]a,b[\neq \emptyset$ and $c \in X$, we have then $X_1 \cap X_2 \cap X \neq \emptyset$, a contradiction.

(2) It suffice to prove that the image by a definable continuous map of a def.-connected set is also def-connected, and this is "the same proof" as for usual connectedness. $\square$

Below, we will use constantly the following basic facts and principles for an o-minimal structure M:

(1) $X \subset M$ definable and <u>infinite</u>

$$\Downarrow$$

$\exists$ o-interval $I \subset X$

(2) $X \subset M$ definable and open

$$\Downarrow$$

X is a finite disjoint union of o-intervals

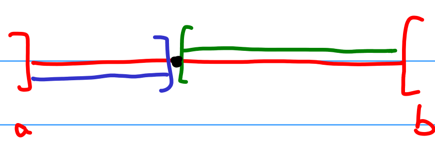
(3) ["selection"] If $I \subset M$ is an o-interval and $I = \bigcup_{i=1}^{n} X_i$, X_i definable and disjoint, then there is a unique i s.t. "X_i contains the right-hand of I", i.e. $\exists\, y \in I$, $]y, \sup(I)[\subset X_i$.

We will say that "I selects X_i (on the right)".

Moreover, if $I \neq X_i$, then there is a <u>smallest</u> $y_0 \in I$ s.t. $]y_0, \sup(I)(\subset X_i$. We then

write $y_0 = sel(I)$,

or $sel_{(X_i)}(I)$.



[<u>Proof</u> - Define i by the condition $\sup(X_i) = \sup(I)$. Then i is unique, and some of the open intervals in X_i must have sup equal to $\sup(I)$.

If $X_i \neq I$, pick $x_0 \in I - X_i$; then all $y \in I$ s.t. $]y, \sup I[\subset X_i$ are $> x_0$; let y_0 be the inf of those y, so $y_0 \geq x_0$. If $y > y_0$ then $\exists z$, $y_0 < z < y$ s.t. $]z, \sup I[\subset X_i$, in particular $y \in X_i$, so $]y_0, \sup I[\subset X_i$.]

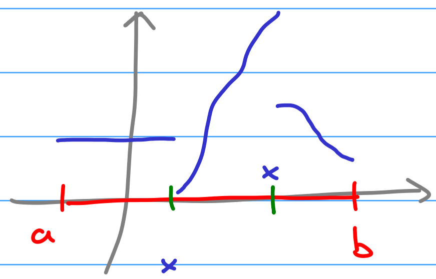
In the next two sections we start studying definable sets in M^m with $m \geq 2$, by beginning with the "simplest" examples in M^2: (1) graphs of functions

(2) sets with "finite fibers" over M

2 - The local structure theorem

Theorem - [vdD Th. 3.1.2, "Monotonicity th."]

> M o-minimal, $I \subset M$ o-interval, $f : I \longrightarrow M$ definable.
>
> There are $a_1, \dots, a_k$ in I s.t., with $a_0 = \operatorname{Inf} I$, $a_{k+1} = \operatorname{Sup} I$, the restriction to $]a_i, a_{i+1}[$ of f is either constant or continuous and strictly monotonic.



To streamline the proof, we will say that a function $f : I \longrightarrow M$ is $\left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\}$-simple if f is continuous and for all $y < z$ in I, we have

$$f(y) \begin{matrix} < \\ = \\ > \end{matrix} f(z).$$

Also f simple means R-simple for one of the three relations.

Then f is continuous and either constant or strictly monotonic iff it is simple.

The proof will require a bunch of lemmas. In all of them, $I \subset M$ is an o-interval and $f : I \longrightarrow M$ is definable.

<u>Lemma 1</u>. (easy) There is $J \subset I$ o-interval s.t. $f|J$ is constant or injective.

<u>Lemma 2</u>. (tricky) If f is injective, there is an o-interval $J \subset I$ s.t. $f|J$ is strictly monotonic.

<u>Lemma 3</u>. (easy) If f is strictly monotonic then there is $J \subset I$ o-interval s.t. $f|J$ is continuous.

<u>Proof of Th. assuming lemmas</u>.

<u>Step 1</u>. Assume that the result holds <u>locally</u>: for any $x \in I$, there is an o-interval $J \subset I$ with $x \in J$ s.t. $f|J$ is simple definable because continuity is

Then I is the union of three disjoint open sets (where f is, resp., $<, =, >$ - simple); since I is d-connected, I is one of them, say the one with $<$.

Let then $x_0 \in I$; then

$$]x_0, b[= \{x \in]x_0, b[\mid f \text{ } <\text{-simple on } [x_0, x[\} \cup (\text{complement}).$$

Each of the two sets is open in $[x_0, b[$ (the first by assumption, the second because if f is <u>not</u> $<$-simple on $[x_0, x[$, then we can find

$x_0 \leq y < z < x$ s.t. $f(y) \geq f(z)$, and this shows that f is not $<$-simple also on $[x_0, w[$ for $z < w < x$), and the first is not empty so in fact f is $<$-simple on $[x_0, b[$.

Similarly, f is $<$-simple on $]a, x_0]$, hence the result.

<u>Step 2</u> - In general, let $X \subset I$ be the set of

$x \in I$ s.t. f isn't simple close to x. Then X is definable, and cannot contain an o-interval J, as we could apply Lemmas 1, 2, 3 to get a contradiction. So X is finite; on each open interval between $-\infty$, successive points of X, $+\infty$, we can apply Step 1.

$\square$

Now we prove the Lemmas 1.

<u>Proof of Lemma 1.</u> [$\exists J \subset I$, $f|J$ injective ou constante]

If there exists $y \in M$ s.t. $f^{-1}(y) \subset M$ is infinite, then (since $f^{-1}(y)$ is definable) $f^{-1}(y)$ contains an o-interval J and $f|J$ is constant.

In the opposite case, $f^{-1}(y)$ is finite for all $y \in M$. The set $f(I)$ is then an <u>infinite</u> (I infinite) definable set; we define

$$g : f(I) \longrightarrow M$$

by $\quad g(y) = \mathrm{Min}\,\overbrace{\{x \in I \mid f(x) = y\}}^{\text{finite},\ \neq \emptyset}.$

The function g is definable and $f \circ g = \mathrm{Id}_{f(J)}$, so g is injective; its image is infinite in I hence contains $J \subset I$ o-interval; then

$$f|J : J \longrightarrow M$$

is injective.

$\square$

<u>Proof of Lemma 3</u>. [f strictly monotone $\Rightarrow \exists\, J \subset I$, $f|J$ cont.]

The function f is then injective, so its image is infinite; let $y_1 < y_2$ in $f(I)$ s.t. $]y_1, y_2[\ \subset f(I)$ and $x_i \in I$ the unique element with $f(x_i) = y_i$. Then $f^{-1}(]y_1, y_2[) =]x_1, x_2[$ (assuming f increasing, otherwise $]x_2, x_1[$; indeed if $y_1 < y < y_2$, then $y = f(x)$ for some unique $x \in I$, and f strictly increasing implies that $x_1 < x < x_2$). Similarly, if $y_1 < t_1 < t_2 < y_2$

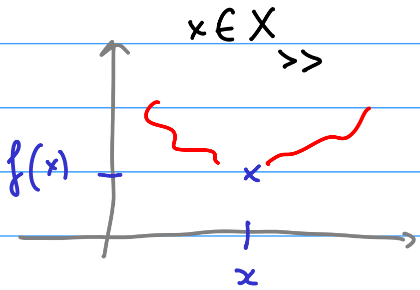
Then $f^{-1}(]t_1, t_2[) =]f^{-1}(t_1), f^{-1}(t_2)[$ so f restricted to $]x_1, x_2[$ is continuous.

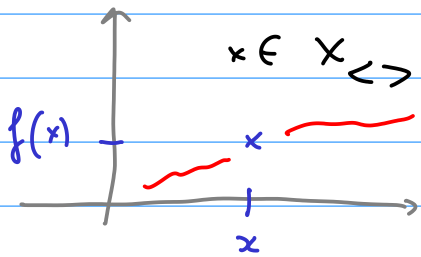
$\square$

<u>Proof of Lemma 2</u>. [f injective $\Rightarrow \exists\, J \subset I$, $f|J$ strictly monotone]

For relations R, S in $<, >$, let

$$X_{RS} = \{ x \in I \mid \exists\ y_1 < x < y_2,\ f(x)\, R\, f(y) \text{ if } y_1 < y < x,\ f(x)\, S\, f(y) \text{ if } x < y < y_2 \}$$





Each X_{RS} is definable in M.

<u>Sublemma 1</u>. If $I = X_{<>}$ or $X_{><}$ then f is strictly monotone on I.

<u>Proof</u>. Suppose $I = X_{<>}$, the other case being similar.

For all $x \in I$, we define

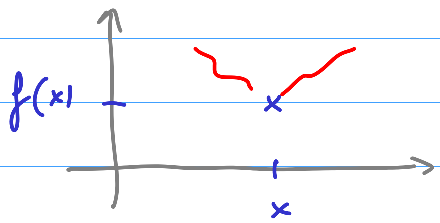
$$s(x) = \underbrace{\sup\{y > x \mid f(z) > f(x) \text{ if } x < y \leq z\}}_{\substack{\text{not empty because } x \in X_{<>} \\ \text{and definable}}}.$$

Then $s(x) = \sup(I)$: otherwise $s(x) \in I$ is a max; since $s(x) \in X_{RS}$, we get $f(y) > f(s(x)) > f(x)$ for $y > s(x)$ close to $s(x)$, contradicting the definition of sup.

But $s(x) = \sup(I)$ means that $f(y) > f(x)$ for all $y > x$. Since x is arbitrary, f is strictly increasing.

$\square$

<u>Sublemma 2</u>. If $I = X_{>>}$ (resp. $X_{<<}$) then $\exists\, J \subset I$, o-interval, s.t. $f|J$ is strictly monotone.



<u>Proof</u>.

Let $X = \{x \in I \mid \forall\, y > x \text{ in } I,\ f(y) > f(x)\}$.

This is a definable set; if it is infinite, it contains an o-interval J and $f|J$ is strictly increasing.

Suppose then X is finite. Let $a = \max(X)$ and $I' =]a, \sup(I)[\subset I$. Replacing I by I', we can assume $X = \emptyset$:

$$(*) \qquad \forall x \in I, \exists y \in I, \; y > x \wedge f(y) < f(x)$$

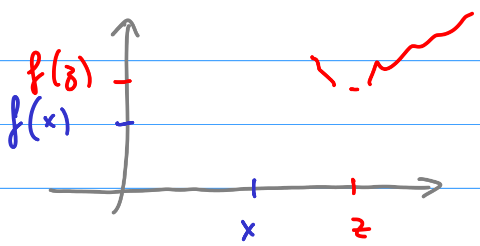
Let then $x \in I$. Since f is injective

$$]x, \sup(I)[= \underbrace{\{ y \in I \mid y > x \text{ and } f(y) > f(x)\}}_{X_1}$$

$$\cup \underbrace{\{ y \in I \mid y > x \text{ and } f(y) < f(x)\}}_{X_2}$$

where each set is definable and they are disjoint.

Suppose the first set is "selected"; since the second is $\neq \emptyset$ by $(*)$, we get a smallest $z \in]x, \sup I[$ s.t. $f(y) > f(x)$ if $y > z$.



Then $f(z) < f(x)$

(otherwise z is not minimal

since $z \in R_{>>}$), but then $(*)$ gives $y > z$

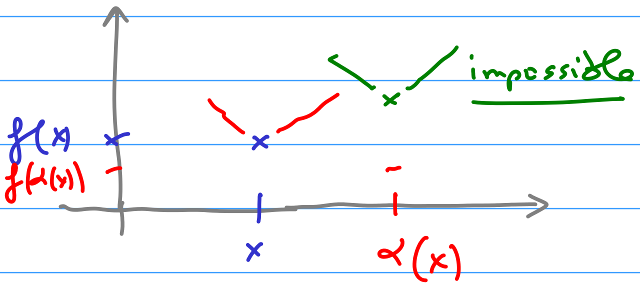
such that $f(y) < f(z) < f(x)$: contradiction!

So the second set X_2 is selected; from $x \in X_{>>}$,

we see that $]x, \sup(I)[\neq X_2$. Let then

let $\alpha(x) \in]x, \sup(I)[$ be minimal s.t.

$$(**) \qquad f(y) < f(x) \quad \text{if} \quad y > \alpha(x).$$



We get $f(\alpha(x)) < f(x)$

from $\alpha(x) \in X_{>>}$.

$$\text{Then} \quad]x, \alpha(x)[= \{ y \mid f(y) > f(x) \}$$
$$\cup \{ y \mid f(y) < f(x) \}.$$

Both sets are definable; the second $(f(y) < f(x))$

cannot be selected since this would contradict

the minimality of $\alpha(x)$, so there is $z < \alpha(x)$

s.t. $f(y) > f(x)$ if $z < y < \alpha(x)$.

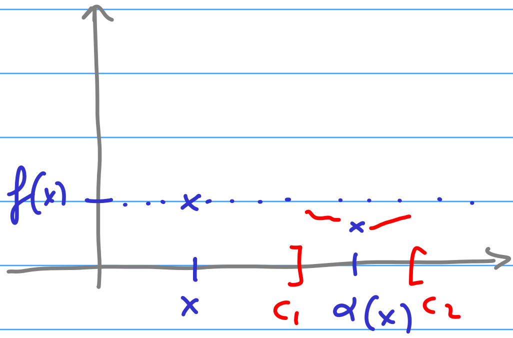
The conclusion is that the formula

$$\phi(v) : \exists c_1 \exists c_2 \; \Big(c_1 < v < c_2 \wedge \big(c_1 < y_1 < v < y_2 < c_2 \longrightarrow f(y_1) > f(y_2) \big)\Big)$$

is satisfied for $v = \alpha(x)$, so the sentence

$$\forall x, \; \exists y > x, \quad \phi(y)$$

is satisfied.



So $\Phi(M)$ is infinite,

so $\Phi(M)$ contains an o-interval J.

But arguing mutatis mutandis in J, we get

there an infinite set satisfying

$$\tilde{\phi}(v) : \exists c_1 \exists c_2 \; \Big(c_1 < v < c_2 \wedge \big(c_1 < y_1 < v < y_2 < c_2 \longrightarrow f(y_1) < f(y_2) \big)\Big)$$

and this is impossible since $\phi \wedge \tilde{\phi}$ cannot be satisfied.

$\square$

Now we can finally conclude!

First, let $x \in I$; then (again)

$$]x, \sup(I)[= \{ \underset{y<x}{y>} x \mid f(y) < f(x) \}$$

$(]\operatorname{Inf}(I), x[)$

$$\cup \{ \underset{y<x}{y>} x \mid f(y) > f(x) \}$$

and one set is selected from both, showing

that $I = X_{<<} \cup X_{<>} \cup X_{><} \cup X_{>>}$

which is a disjoint union of definable sets.

So one contains an o-interval J and we

conclude by applying sublemma 1 or 2 to

J (the former if $J \subset X_{<>}$ or $X_{><}$,

the latter otherwise).

This concludes the proof of the theorem!

<u>Corollary</u> - (1) $f : I \longrightarrow M$ definable

$\Rightarrow \forall a \in I$, $\lim\limits_{\substack{x \to a \\ x<a}} f(x)$, $\lim\limits_{\substack{x \to a \\ x>a}} f(x)$ exist

in M and $\lim\limits_{x \to \sup(J)} f$, $\lim\limits_{x \to \operatorname{Inf}(a)} f$ exist in

$\overline{M}$.

(2) $I = [a, b]$, $f : I \longrightarrow M$ definable and continuous

$\Rightarrow f$ has max. and min.

3. Uniform finiteness

Th. Let M be o-minimal and $X \subset M^2$ definable such that

$$\forall a \in M, \quad X_a = \{ b \in M^2 \mid (a,b) \in X \}$$

is finite. There exist

$$-\infty = a_0 < a_1 < \dots < a_k < a_{k+1} = +\infty$$

and for $0 \leq i \leq k$, there exists $c_i \geqslant 0$

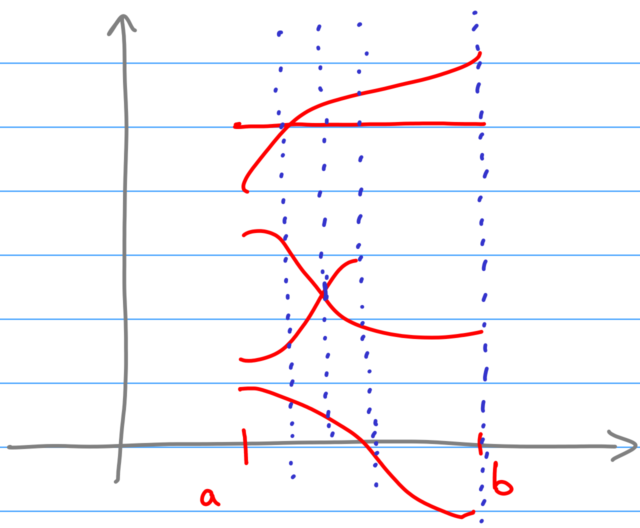
and $\quad f_{i,1}, \dots, f_{i,c_i} :]a_i, a_{i+1}[\longrightarrow M$

definable and continuous such that

$$\forall a \in]a_i, a_{i+1}[, \quad f_{i,1}(a) < \dots < f_{i,c_i}(a)$$

and $\quad X_a = \{ f_{ij}(a) \mid 1 \leq j \leq c_i \}.$

In particular: the cardinality of X_a is uniformly bounded.



$c_1 = c_2 = c_3 = 5$

$c_4 = 4$

Remark. The final consequence is understandable for models of RCF, but the precise statement is not obvious!

Proof. Let $c(a) = |X_a|$ for $a \in M$ and write $y_1(a) < \dots < y_{c(a)}(a)$ the elements of $c(a)$ in increasing order (allowing $c(a) = 0$ of course).

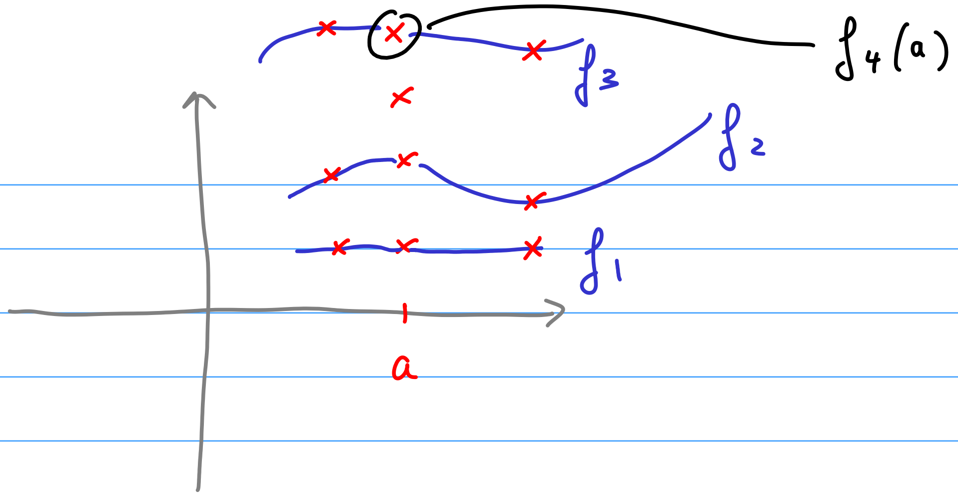
Let $D_n = \{ a \in M \mid c(a) \geq n \}$

$$f_n : \begin{cases} D_n \longrightarrow M \\ a \longmapsto y_n(a) \end{cases}$$

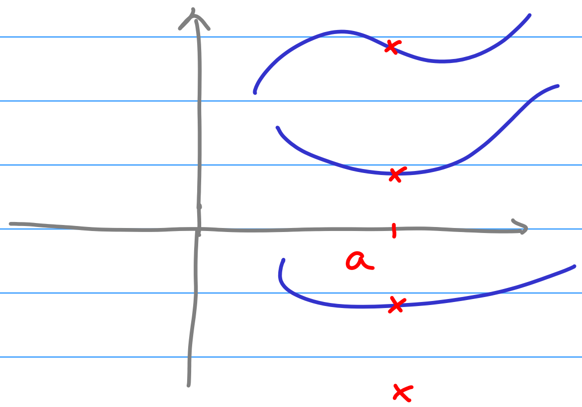
For each $n \geq 0$, D_n and f_n are definable; also $D_{n+1} \subset D_n$

and $f_{n+1}(a) > f_n(a)$ if $a \in D_{n+1}$.

Further: let $\tilde{c}(a) \leq c(a)$ be the largest integer s.t. there is an o-interval I containing a such that $f_1, \dots, f_{\tilde{c}(a)}$ are defined and continuous on I.



$c(a) = 4$, $\tilde{c}(a) = 2$

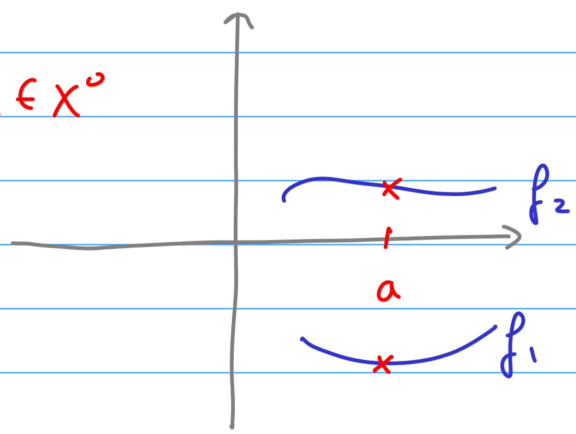


$c(a) = 4$, $\tilde{c}(a) = 0$

[Not definable, a priori]

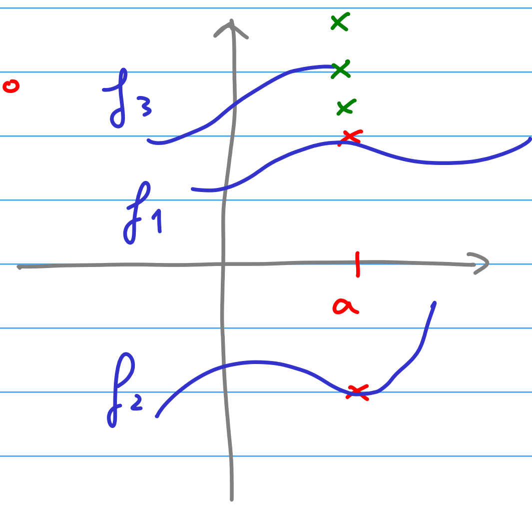
Finally let $X^\circ = \{ a \in M \mid a \notin \overline{D_{\tilde{c}(a)+1}} \}$.

$a \in X^\circ$



$\tilde{c}(a) = 2 = c(a)$

$a \notin X^\circ$



$\tilde{c}(a) = 2 < c(a)$ (if some green points are there)

Lemma. The maps $a \mapsto c(a)$ and $a \mapsto \tilde{c}(a)$ are locally constant on X°, and equal.

Proof – Let $a \in X^\circ$ so $a \notin \overline{D_{\tilde{c}(a)+1}}$;

there is an open interval $I \ni a$ such that

$I \cap D_{\tilde{c}(a)+1} = \emptyset$; this means that

for $b \in I$, we have $c(b) \leq \tilde{c}(a)$;

since on an $I' \subset I$ open containing a

the functions $f_1, \dots, f_{\tilde{c}(a)}$ are defined and

continuous we have

$$X_b = \{ f_1(b), \dots, f_{\tilde{c}(a)}(b) \}$$

there, so $c(b) = \tilde{c}(b) = \tilde{c}(a) = c(a)$ on I'.

$\square$

The key lemma is the following:

Lemma – $M - X^\circ$ is a finite set.

Assuming this, we conclude easily: let

$$-\infty = a_0 < \underbrace{a_1 < \dots < a_k}_{\text{the elements of } M - X^\circ} < a_{k+1} = +\infty$$

Let $0 \leq i \leq k$; on $]a_i, a_{i+1}[\subset X^\circ$, the

functions c, $\tilde{c}$ are locally constant and

equal with definable level sets ; by d-con-
-nectedness, they are equal and constant, equal to c_i, say. Then for $a \in \,]a_i, a_{i+1}[$, we have

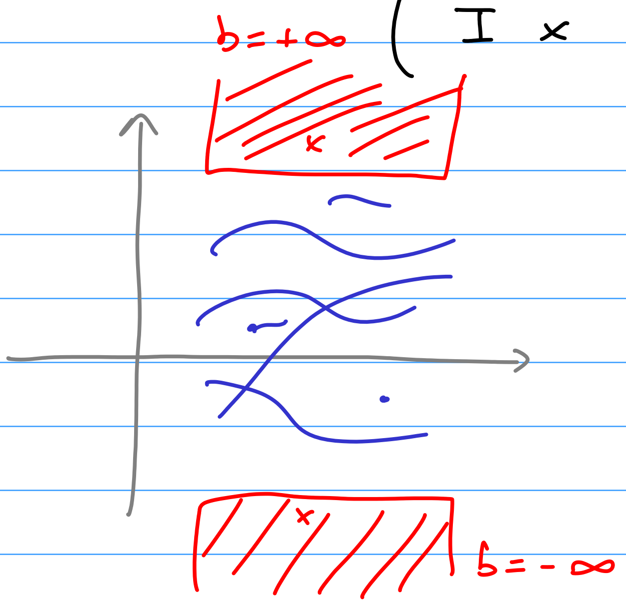
$$X_a = \{ f_1(a), \dots, f_{c_i}(a) \}$$

so we get the result with $f_{i,j}$ = restriction of f_i to $]a_i, a_{i+1}[$.

Proof of the lemma – Let $N \subset M \times \overline{M}$ the set of $(a,b) \in M^2$ s.t.

(i) if $b = \pm\infty$, there is an o-interval $I \ni a$, $y \in M$ s.t.

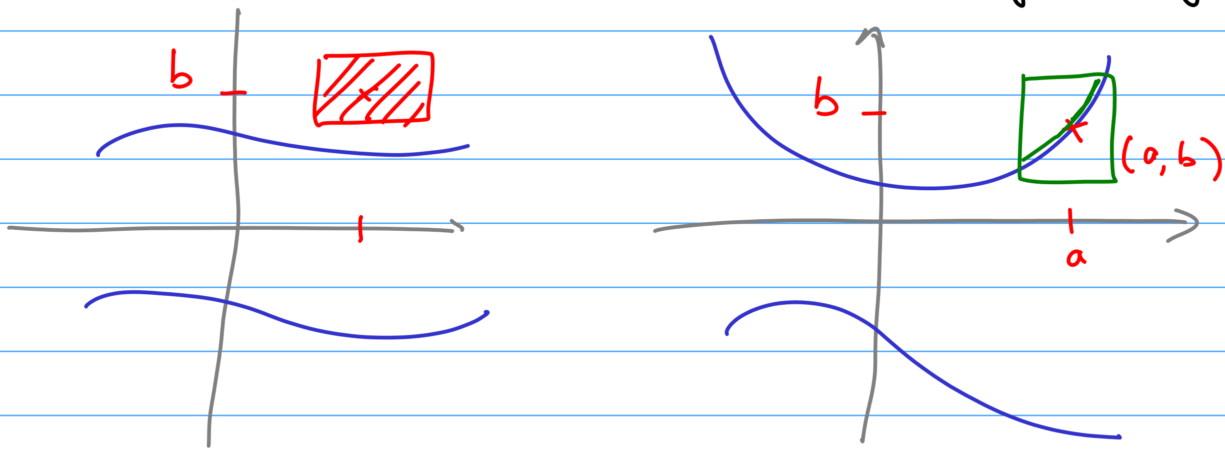
$$\left(I \times \begin{matrix}]y, +\infty[\\ ]-\infty, y[\end{matrix} \right) \cap X = \emptyset$$



(ii) if $b \in M$, either $\exists\, I \ni a$, J o-interval containing b s.t.

$$(I \times J) \cap X = \emptyset,$$

$\boxed{\text{or}}$ $\exists\ I, J$ as above and $f : I \longrightarrow M$ continuous with $(I \times J) \cap X =$ graph of f.



This set N is definable (with an obvious meaning as far as $(a, \pm\infty) \in N$ definably means ...) and we will show that

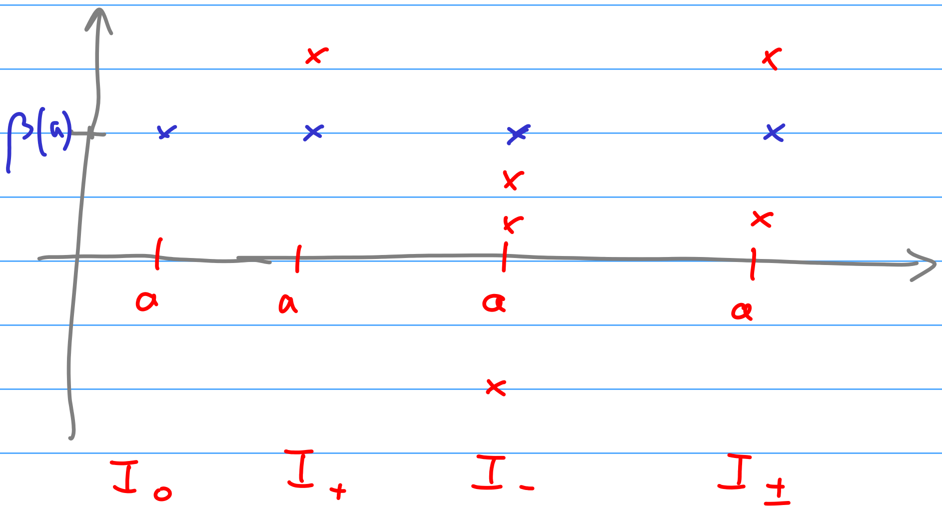
$\circledast$ $X^\circ = \{ a \in M \mid \forall b \in \overline{M},\ (a,b) \in N \}$

which then implies that X° is definable.

Assuming $\circledast$, we then see that if $M - X^\circ$ is not finite, it contains an o-interval I. We can then conclude as follows: for $a \in I$, the set of $b \in \overline{M}$ s.t. $(a,b) \notin N$ is definable, closed, non empty, so has a smallest element $\beta(a) \in \overline{M}$, and $\beta : I \longrightarrow \overline{M}$ is definable.

Further I is the disjoint union of four definable sets I_0, I_+, I_-, $I_\pm$, defined by the condition that

$$\begin{cases} X_a \subset \{(a, \beta(a))\} \Rightarrow a \in I_0 \\ X_a \cap \,]\beta(a), +\infty[\neq \emptyset \text{ only} \Rightarrow a \in I_+ \\ X_a \cap \,]-\infty, \beta(a)[\neq \emptyset \text{ only} \Rightarrow a \in I_- \\ \text{both} \Rightarrow a \in I_\pm \end{cases}$$



One of these contains an o-interval, say $I_\pm$.

Define
$$\beta_\pm : I_\pm \longrightarrow M$$
$$a \longmapsto \operatorname{Min}\left\{ b \in X_a ,\ \begin{matrix} b > \beta(a) \\ b < \beta(a) \end{matrix} \right\}$$

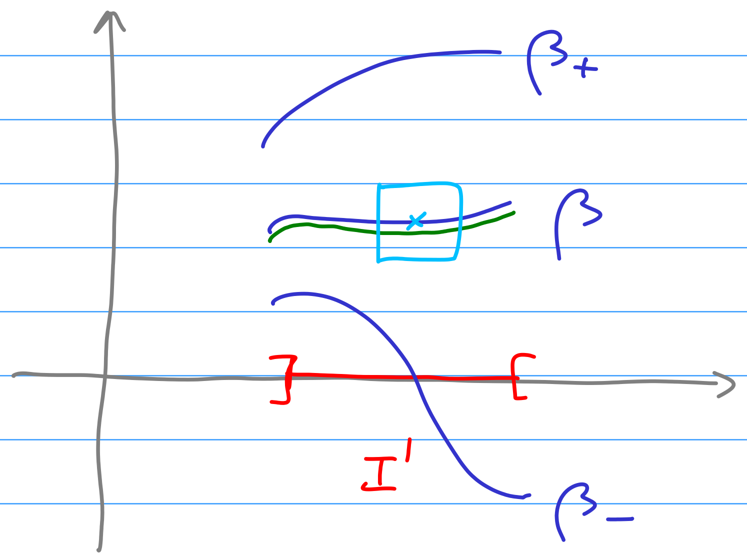
which are both defined / definable on $I_\pm$.

By the Structure Theorem, there is an o-interval $I' \subset I_\pm$ such that β_-, β, β_+ are continuous on I', and by o-minimality, there is an o-interval $I'' \subset I'$ s.t.

<u>either</u> $\forall a \in I'$, $(a, \beta(a)) \in X$

<u>or</u> $\forall a \in I'$, $(a, \beta(a)) \notin X$.

But then it is not difficult to check that $(a, \beta(a)) \in N$ for $a \in N'$, which is a contradiction. The other cases (I_0, I_-, I_+) reach the same conclusion (more easily).



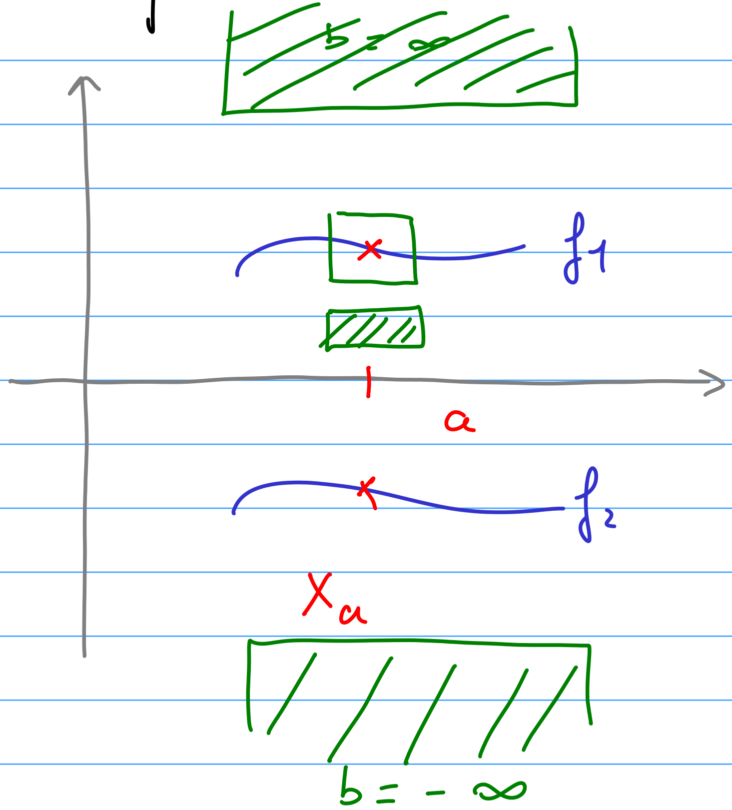
<u>Case 1</u>: $(a, \beta(a)) \in X$ $\forall a \in I'$

$\Rightarrow$ $\exists \tilde{I} \ni a, \tilde{J} \ni \beta(a)$, $(\tilde{I} \times \tilde{J}) \cap X = \Gamma_{\beta|\tilde{I}}$

Proof of the Lemma -

If $a \in X^{0}$, then it is easy to see that

$(a, b) \in N$ for $b \in \overline{M}$:



$c(a) = \tilde{c}(a) = 2$

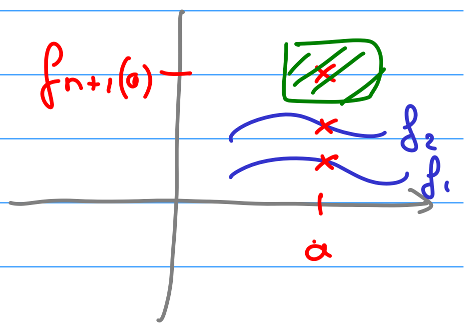
[f_1, f_2 are bounded close to a]

Conversely, supposons $a \notin X^{0}$ so $a \in \overline{D}_{n+1}$

[where $n = \tilde{c}(a)$]. Recall D_{n+1} is definable.

Case 1. $a \in D_{n+1}$ is isolated (some neigh. only intersects D_{n+1} at a); then

$$(a, f_{n+1}(a)) \notin N$$



since $(\tilde{I} \times \tilde{J}) \cap X = \{(a, f_{n+1}(a))\}$

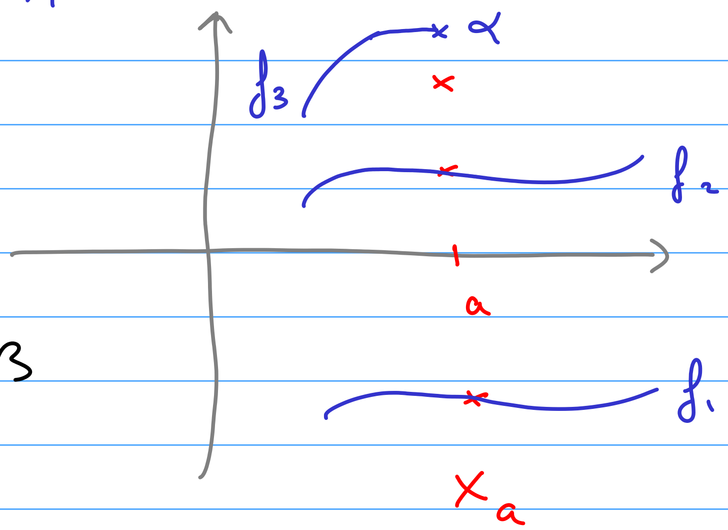
for $\tilde{I}, \tilde{J}$ small enough.

In other cases, D_{n+1} contains an o-interval, $]y, a[$ or $]a, y[$, or both. Let then

$$\alpha = \lim_{\substack{x \to a \\ x < a}} f_{n+1}(x) \quad , \quad \beta = \lim_{\substack{x \to a \\ x > \beta}} f_{n+1}(x)$$

when these exist (at least one does).

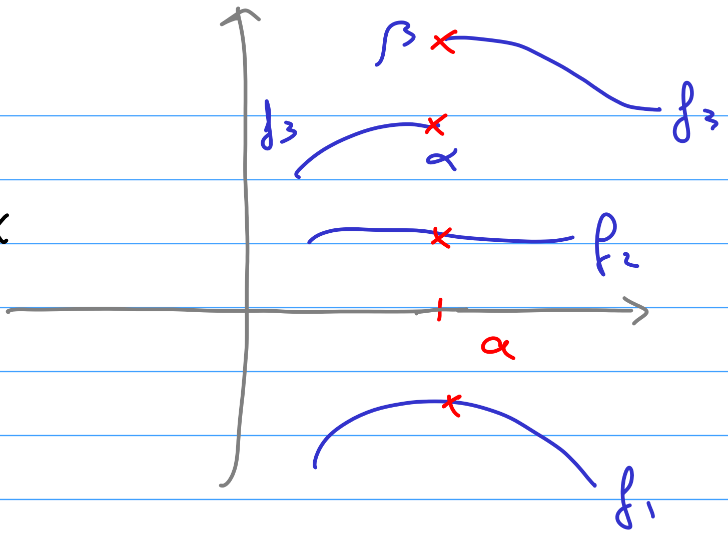
Case 2 - If (a, α) (resp. (a, β)) exists but is not in X, then $(a, \alpha) \notin N$ (resp. $(a, \beta) \notin N$): (including $\alpha = \pm\infty, \beta = \pm\infty$)



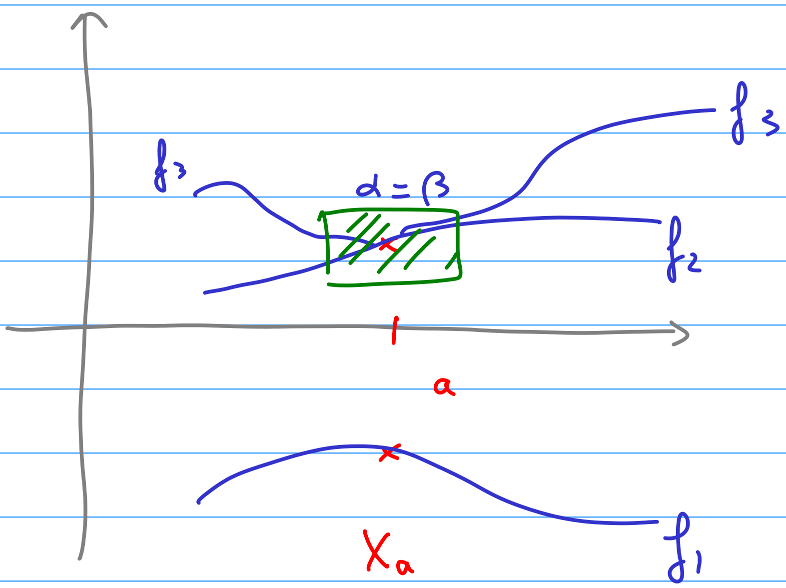
either there is only one of α, β defined, then $(\tilde{I} \times \tilde{J}) \cap X$ is "half a curve", or both are defined but $(\tilde{I} \times \tilde{J}) \cap X$ "misses the value at a".

Case 3 - α, β are both defined and are in X and $\alpha \neq \beta$ then $(a, \alpha) \notin N$

(and $(a,\beta) \notin N$) :
the intersections $(\tilde{I} \times \tilde{J}) \cap X$
only give "half a graph".



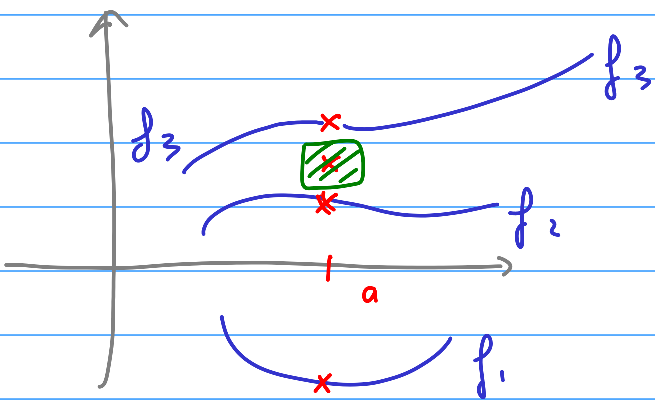
Case 4. α, β are defined, $\alpha = \beta \in X$.



So $(a, \alpha) \in X_a$;
since $f_{n+1}(x) \geq f_n(x)$
for all x, we
have $\alpha \geq f_n(a)$.

If $\alpha = f_n(a)$, then $(a, \alpha) \notin N$: there are
two curves converging to (a, α)

Otherwise, $a \in D_{n+1}$. If $\alpha \neq f_{n+1}(a)$,



then $(a, f_{n+1}(a)) \notin N$
(it is isolated in X)
and (finally) if

$\alpha = f_{n+1}(a)$, then this means that f_{n+1} is defined and continuous at a, which contradicts $n = \tilde{c}(a)$.

$\square$