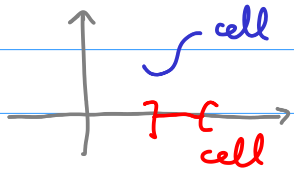
Chapter VI

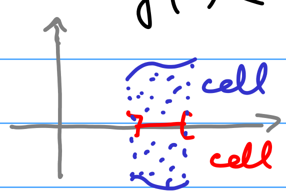
Cellular decomposition

The goal is to find a "good" geometric description of all definable sets in an o-minimal structure. Throughout: M is o-minimal.

1 - Cells

The building blocks of all definable sets will be cells. These are subsets obtained by, inductively, either (1) taking the graph of a continuous function defined on a previously defined cell, or (2) taking the "space" between two continuous functions f, g defined on a previous cell, with $f(x) < g(x)$ on that cell.





For a precise definition, let $\mathbb{B}_m = \{0, 1\}^m$ for $m \geqslant 0$ and $\mathbb{B} = \bigcup_m \mathbb{B}_m$, with $|\sigma| = m$

if $c \in \mathbb{B}_m$. Define also $c' = (c_1, \dots, c_{m-1})$ if $c = (c_1, \dots, c_m) \in \mathbb{B}_m$, $m \geq 1$.

<u>Definition</u>. For $c \in \mathbb{B}$, the set $\mathrm{Cells}_c(M)$ of c-cells are defined as follows:

(1) $\mathrm{Cells}_{()}(M) = \{M^0\}$

$()$ – unique element of $\mathbb{B}_0$

(2) If $C' \in \mathrm{Cells}_{c'}(M)$ and $f : C' \longrightarrow M$ is continuous then $\Gamma_f \in \mathrm{Cells}_{(c',0)}(M)$.

(3) If $C' \in \mathrm{Cells}_{c'}(M)$ and $f, g : C' \longrightarrow \overline{M}$ are continuous and $f(x) < g(x)$ for all $x \in C'$, then $]f, g[\in \mathrm{Cells}_{(c',1)}(M)$

were $]f, g[= \{(x,y) \in C' \times M \mid f(x) < y < g(x)\}$.

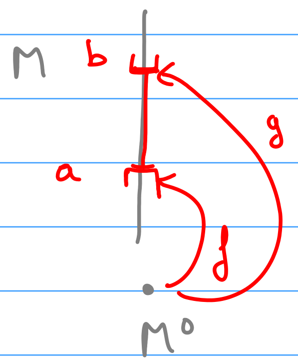
An M-cell is an element of $\mathrm{Cells}_c(M)$ for some $c \in \mathbb{B}$. It is then a definable subset of $M^{|c|}$.

<u>Remark</u>. If $f : X \longrightarrow \overline{M}$ is continuous, with X d-connected, then either $f(X) \subset M$

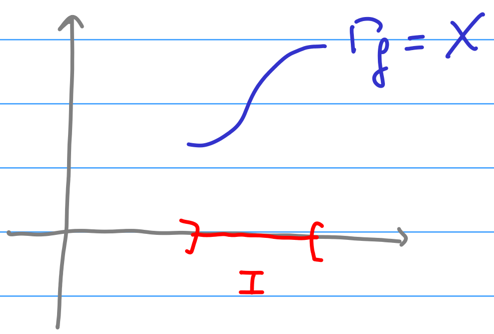
or $f = +\infty$ (constant) or $f = -\infty$.

Examples. (1) $\mathrm{Cells}_{(0)}(M) = \{\{a\} \mid a \in M\}$

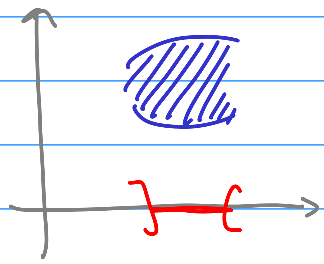
(since $\mathrm{Cells}_{()}(M) = \{M^0\}$ and M^0 has a single point); $\mathrm{Cells}_{(1)}(M) = \,]a, b[$ with $a < b$ in $\overline{M}$ (same reason).



(2) $X \in \mathrm{Cells}_{(1,0)}(M) \iff X$ is the graph of a function $f : I \longrightarrow M$, I o-interval in M, continuous,



And $X \in \mathrm{Cells}_{(1,1)}(M)$ has the form



On the other hand, $X \in \mathrm{Cells}_{(0,0)}(M)$ is a single $(a, b) \in M^2$, and $X \in \mathrm{Cells}_{(0,1)}(M)$ is a vertical segment $\{(a,b) \in M^2 \mid b_1 < b < b_2\}$.

So for instance: if $f, g : [a_1, a_2] \longrightarrow M$ are continuous then $\{ (a,b) \in M^2 \mid a_1 \leq a \leq a_2$ and $f(a) \leq b \leq g(a) \}$

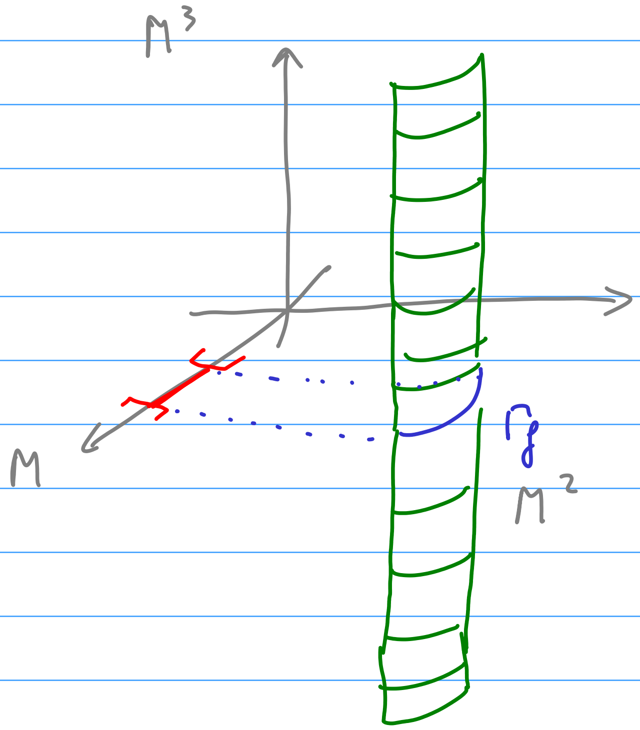
is the disjoint union of

(1) two $(0,1)$-cells $\big[\{ (a_1, b) \mid f(a_1) < b < g(a_1) \}$ and $\{ (a_2, b) \mid f(a_2) < b < g(a_2) \} \big]$

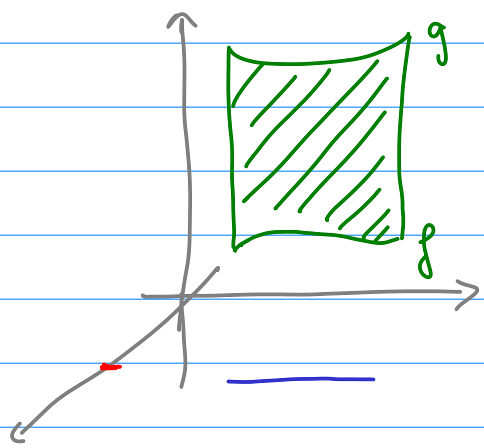
(2) two $(1,0)$-cells $\big[\Gamma_{f |]a_1, a_2[}$ and $\Gamma_{g |]a_1, a_2[} \big]$

(3) one $(1,1)$-cell $]f|]a_1,a_2[\, , \, g|]a_1,a_2[[$

(3) $X \in \mathrm{Cells}_{(1,0,1)}(M)$ means that $X \subset M^3$ is a kind of "piece of half-cylinder": e.g.



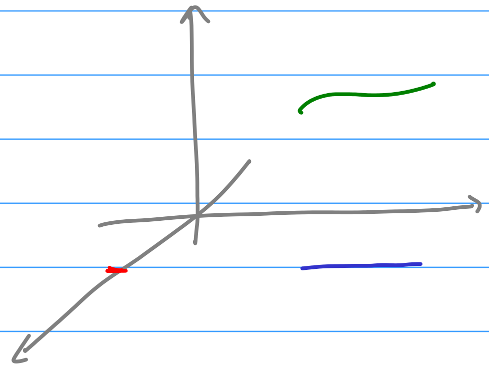
$X \in \mathrm{Cells}_{(0,1,1)}(M)$:



$\text{Cells}_{(0,0,0)}(M)$: single $(a,b,c) \in M^3$

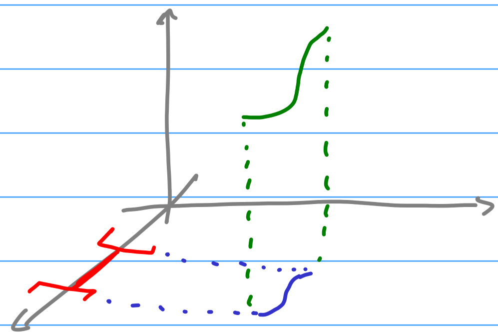
$\text{Cells}_{(0,0,1)}(M)$: segment (a,b,c), $c_1 < c < c_2$

$\text{Cells}_{(0,1,0)}(M)$:



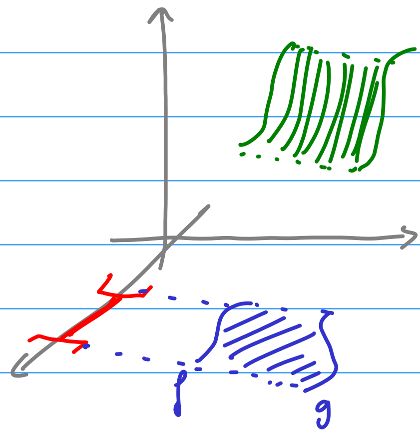
$$\{ (a, b, f(b)) \mid b_1 < b < b_2 \}$$

$\text{Cells}_{(1,0,0)}$:



$$\{ (a, f(a), \tilde{f}(a)) \mid a_1 < a < a_2 \}$$

$\text{Cells}_{(1,1,0)}$:



$$\{ (a, b, f(a,b)) \mid a_1 < a < a_2, \; f(a) < b < g(a) \}$$

$\text{Cells}_{(1,1,1)}$: (some) open subsets of M^3

$(m \geq 1)$

Notation. $\pi_m : M^m \longrightarrow M^{m-1}$

$(x_i) \longmapsto (x_1, \ldots, x_{m-1})$

$X \subset M^m \rightsquigarrow \pi_m(X) = X' \subset M^{m-1}$

If $C \in \mathrm{Cells}_c(M)$ then $C' \in \mathrm{Cells}_{c'}(M)$

[and C is either Γ_f or $]f, g[$ for suitable continuous functions].

2. Cellular decompositions

Definition. $X \subset M^m$.

(1) A cellular partition of X is a finite (possibly empty!) partition $(C_j)_{j \in J}$ of X s.t. each C_j is a cell.

(2) A cellular decomposition of X is a cellular partition $(C_j)_{j \in J}$ s.t.

$\forall k, \; 1 \leq k \leq m, \; \{C_j^{(k)}\}$ is a cellular partition of $X^{(k)} = \underbrace{((X')')\ldots)'}_{k \text{ times}} \subset M^{m-k}$.

Note. In fact, $1 \leq k \leq m-1$ suffices since $k = m$

is trivial.

Example. (1) "Every $X \subset M$ has a cellular decomposition" is equivalent to the definition of o-minimality.

(2) If $X \subset M^2$ and all fibers X_a are finite, then X has a cellular decomposition with cells of type $(0,0)$ or $(1,0)$. (This exactly the "Uniform Finiteness" Theorem.) This applies in particular to Γ_f for $f: I \longrightarrow M$ definable (which also follows from the local structure theorem).

Theorem. (Cellular decomposition)

Let $X \subset M^m$ be definable. Then X has cellular decompositions.

More precisely, the following statements are true:

(1) For any finite family $(X_i)_{i \in I}$ of definable subsets of X, there is a cellular decomposition $\mathcal{D}$ which is adapted: each X_i is a union [disjoint] of some cells of $\mathcal{D}$.

(2) For any $n \geq 0$ and $f : X \longrightarrow M^n$ definable, there is a cellular decomposition $\mathcal{D}$ such that $f|C$ is continuous for alls cells C of $\mathcal{D}$.

Before going to the proof, we will gather some facts about cells and derive corollaries of the theorem.

3 _ Basic properties of cells

Proposition. M o-minimal

(1) Each cell is locally closed [= intersection of an open set and a closed set = C is open in its closure $\overline{C}$] in $M^{|C|}$

(2) A cell $C \in \mathrm{Cells}_c(M)$ is open (in $M^{|c|}$) $\iff$ $c = (1, \dots, 1) = 1_m$. If $c \neq 1_m$, then $\mathring{C} = \emptyset$.

(3) Any cell is d-connected.

<u>Proof.</u> (1) exercise

(2) If C is open then C' also (π_m is an open map) so by induction we may assume that $c' = 1_{m-1}$.
If $c = (1_{m-1}, 0)$, then $C = \Gamma(f)$ for some continuous definable $f : C \to M$. But then $\mathring{C} = \emptyset$. So we must have $c = (1_{m-1}, 1) = 1_m$. And then

$$C = \left] f, g \right[$$

for $f, g : C' \longrightarrow M$ continuous and definable.
It is then elementary that C is indeed open.

(3) Again we use induction on m. For $c = (c', 0)$, the restriction of π_m to C is a definable homeomorphism $C \longrightarrow C'$ (with inverse $a \mapsto (a, f(a))$ if $C = \Gamma(f)$)

so induction gives the result. For $c = (c', 1)$, note that C' is d-connected by induction. Let $f, g : C' \longrightarrow M$ be continuous and definable so that $C =]f, g[$. Suppose $C = U_1 \cup U_2$ with U_i definable, open, disjoint. For $x \in C'$, we get $]f(x), g(x)[= \big(\underbrace{U_1 \cap (\{x\} \times M)}_{W_1(x)}\big) \cup \big(\underbrace{U_2 \cap \{x\} \times M}_{W_2(x)}\big)$ and since o-intervals are connected, either $W_1(x)$ or $W_2(x)$ is $]f(x), g(x)[$. Let $V_i \subset C'$ be the set of those $x \in C'$ s.t. $W_i(x) =]f(x), g(x)[$. This is definable, $V_1 \cap V_2 = \emptyset$ and (because U_i is open) we get W_i open, so (since C' is d-connected) either $C' = V_1$ or $C' = V_2$, which gives $U_1 = C$ or $U_2 = C$.

$\square$

Also using induction, one proves:

Proposition. Let $C \in \mathrm{Cells}_c(M)$, let

$$I = \{ i_1 < \cdots < i_k \} \quad , \quad k \leq |c|$$

be the set of $1 \leq i \leq |c|$ s.t. $c_i = 1$. Let

$$p_c : M^{|c|} \longrightarrow M^{|I|}$$

be defined by $(x_i) \longmapsto (x_{i_1}, \ldots, x_{i_k})$.

Then $p_c|C : C \longrightarrow M^{|I|}$ is a definable homeomorphism on its image, which is an <u>open</u> cell in $M^{|I|}$.

4. <u>Applications of the cellular decomposition theorem</u>

<u>Proposition 1</u>. Let $m \geq 1$ and $X \subset M^m$ definable. Suppose $\pi_m^{-1}(a)$ is finite for all $a \in M^{m-1}$. Then there exists $N \geq 0$ s.t. $|\pi_m^{-1}(a)| \leq N$ for all a.

This is because the assumption implies that in a decomposition $\mathcal{D}$ of X, we can only have cells of type $(c', 0)$. Then we can take N to be the number of cells in $\mathcal{D}$. $\square$

Remark. In fact, a critical step of the proof of the Decomposition Theorem is a direct proof of Prop. 1...

Proposition 2. Let $X \subset M^m$, $m \geq 1$, be definable and $1 \leq n < m$. Let $\pi : M^m \longrightarrow M^n$ be the projection $(x_i) \longmapsto (x_1, \dots, x_n)$.
There exists an integer $N \geq 0$ such that, for all $x \in M^n$, the set $\pi^{-1}(x) \cap X$ has $\leq N$ d-connected components [= maximal d-connected sets] which partition it and are open and closed in $\pi^{-1}(x) \cap X$.

Proof. We assume $n = 0$ for simplicity.
Let $\mathcal{D} = (C_j)_{j \in J}$ be a cellular decomposition of X. For any subset $K \subset J$, let $C_K = \bigcup_{j \in K} C_j$; these are definable subsets of X.
Let $\mathcal{Y}$ be the set of $K \subset J$ such that

C_K is d-connected, and K is maximal for this property.

<u>Claim:</u> The sets C_K, $K \in \mathcal{Y}$, gives the desired d-connected components.

To see this, note first that since each C_j is d-connected, each $K \in \mathcal{Y}$ is non-empty, and each C_j is contained in some C_K, $K \in \mathcal{Y}$ which is moreover <u>unique</u> (if $C_j \subset C_{K_1} \cap C_{K_2}$, then $C_{K_1} \cup C_{K_2} = C_{K_1 \cup K_2}$ is d-connected so $K_1 \cup K_2$ cannot be bigger than K_1 or K_2).

So the C_K, $K \in \mathcal{Y}$, form a finite partition of X in d-connected sets.

There remains to show that each C_K is maximal d-connected and open / closed in X.

Let $x \in \overline{C_K}$; then $C_K \cup \{x\}$ is d-connected, hence $C_K \cup C_j = C_{K \cup \{j\}}$ also if $x \in C_j$.

So $K \cup \{j\} = K$ by maximality, giving $x \in C_K$.

Now since $\mathcal{Y}$ is finite each C_K is also closed $\left(X - C_K = \bigcup_{\substack{L \in \mathcal{Y} \\ L \neq K}} C_L \right)$.

Finally, let $D \subset X$ be d-connected and K such that $D \cap C_K \neq \emptyset$. Then $D \subset C_K$ because $D = \underbrace{(D \cap C_K)}_{\substack{\text{open} \\ \text{in } D}} \cup \underbrace{(D \cap (X - C_K))}_{\text{open in } D}$

so by d-connectedness of D, we must have $D \cap C_K = D$. This implies that each C_K, $K \in \mathcal{Y}$, is maximal d-connected: if $D \supset C_K$ is d-connected, then $D \cap C_K \neq \emptyset$ ...

$\square$

Proposition 3 - Let M' be an $\mathcal{L}$-structure, model of $Th(M)$ ($\Leftrightarrow$ every sentence φ that holds in M also holds in M'). Then M' is o-minimal.

Note. One also says that any o-minimal structure is strongly o-minimal: if two structures are elementarily equivalent, then either none or both is o-minimal.

Proof. Let $\varphi(\underline{y}, x)$ be an $\mathcal{L}$-formula in $m+1$ variables; let $\underline{b}' \in (M')^m$ and

$$X' = \varphi(\underline{b}', M') \subset M'$$

(general definable set with parameters).

We need to check that X' is a finite union of points and intervals.

Now it suffices to show: « $\overline{X'} - \mathring{X}'$ is finite, and if

$$-\infty = a_0 < a_1 < \dots < a_k < a_{k+1} = +\infty$$

are the points of $\overline{X'} - \mathring{X}'$, then for each i, the interval $]a_i, a_{i+1}[$ is contained in either X' or $M' - X'$. »

interval property

What we just wrote is, for each given integer $k \geq 0$, expressible in first order language, <u>uniformly</u> in $\underline{b}'$: there exist formulas $\psi_k(\underline{y})$ such that

$$M' \models \psi_k(\underline{b}')$$
$$\Updownarrow$$
$$\varphi(\underline{b}', M') \text{ has } k \text{ boundary points}$$

and has the "interval property"

Now the key point is that if $\psi(\underline{y}, x)$ is a formula with $\psi(\underline{b}, M) = \text{bdry of } \varphi(\underline{b}, M)$

Then

$$X = \{(\underline{b}, a) \in M^{m+1} \mid \psi(\underline{b}, a)\}$$

contains only finitely many $(\underline{b}, a)$ for given $\underline{b}$, by <u>o-minimality of M</u>. So there is some N such that $|\text{bdry}(X_{\underline{b}})| \leq N$ for <u>all</u> $\underline{b} \in M^m$. But then

$$M \models \forall \underline{y}, \left(\psi_0(\underline{y}) \vee \dots \vee \psi_N(\underline{y})\right)$$

and since this is a <u>sentence</u> now, we get

by assumption $M' \models \forall \underline{y}, (\psi_0(\underline{y}) \vee \dots \vee \psi_N(\underline{y}))$

which in turn implies that $X' = \varphi(\underline{b}', M')$

is indeed a finite union of points and

intervals.

$\square$

5 - <u>Proof of the Theorem</u>

We proceed by induction to prove the following

statements:

$(A)_m$ For any finite family of def. subsets of M^m,
$(m \geq 0)$
there is a cellular decomposition (c.d. from now

on) s.t. all Y in the family are adapted

$(B)_{m,k}$ For any $X \subset M^m$ def. and $f: X \to M^k$
$(m \geq 0, k \geq 0)$
def., there is a c.d. $\mathcal{D}$ of X s.t. $f|C$ is

continuous for all cells C of $\mathcal{D}$.

$(C)_m$ Any def. $X \subset M^m$ (quasi-) finite over
$(m \geq 1)$

M^{m-1} is uniformly (quasi-) finite.

(We used the following terminology: $X \subset M^n$ is *finite* over M^{m-n} if the fibers of the projection $X \longrightarrow M^{m-n}$ are finite, and *unif^ly finite* if they have bounded size.)

The induction proceeds as follows:

start

- $(A)_0, (B)_0, (C)_1$: immediate
- $(A)_1$: definition of o-minimality
- $(B)_{1,1}$: follows from local structure theorem
- $(C)_2$: " " uniform finiteness

Then we show:

Step 1 - $(A)_m + (B)_{m,1} \Longrightarrow \overbrace{(B)_{m,k} \text{ for all}}^{(B)_m}$ k.

Step 2 - $m \geq 2$, $(A)_n, (B)_n, (C_n)$ for $n \leq m-1$ imply $(C)_m$.

Step 3 - $m \geq 2$, $(A)_n$, $(B)_{n,k}$ for $n \leq m-1$ (and $k \geq 0$) and $(C)_m$ imply $(A)_m$.

__Step 4.__ $m \geq 2$, $(A)_n$, (B_n) for $n \leq m-1$ imply $(B)_m$.

__Proof of Step 1.__ Let $f : X \longrightarrow M^k$ be def.; write $f = (f_1, \dots, f_k)$. By $(B)_{m,1}$, find $\mathcal{D}_i$ s.t. f_i is continuous on cells of $\mathcal{D}_i$; then find a c.d. $\mathcal{D}$ of M^m s.t. all cells of all $\mathcal{D}_i$ are adapted to $\mathcal{D}$ (by $(A)_m$). Then __all__ $f_i | C$ are continuous for all cells of $\mathcal{D}$.

$\square$

__Proof of Step 2.__ $[(A)_{<m}, (B)_{<m}, (C)_{<m} \Rightarrow (C)_m]$

We will use some preliminary definitions and statements of purely topological nature.

__Def.__ $X \subset M^m$ def., $p : X \longrightarrow M^{m-1}$ projection

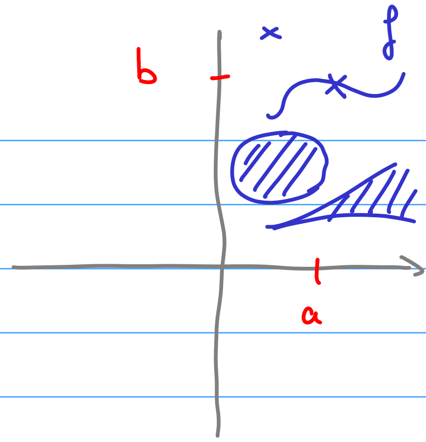
(1) X is __étale__ at $(a, b) \in X$ if:

$\exists\, U$ neighb. of a in M^{m-1}

$\exists\, I$ open interval containing b

$\exists \ f : I \longrightarrow M$ continuous

s.t. $\Gamma(f) = (U \times I) \cap X$



(2) Let $V = X' \subset M^{m-1} = p(X)$;

X is <u>uniformly étale</u> (over V) if one can take $U = V$ in (1) for all $(a,b) \in X$.

(3) X is <u>locally uniformly étale</u> if for all $a \in M^{m-1}$, one can find a neigh. V of a s.t.

$$X_V = X \cap (V \times M) \longrightarrow V \text{ is uniformly}$$

étale.

The following is the basic topological property we need:

<u>Proposition</u> – If $X \subset M^m$ is finite over $B \subset M^{m-1}$ where B is $\left\{\begin{array}{l}\text{locally } d\text{-connected} \\ d\text{-connected}\end{array}\right.$ and if $X \longrightarrow B$ is loc. unif. étale then $X \longrightarrow B$ is "trivial": there is $k \geq 0$ and a homeomorphism

$$B \times \{1, \dots, k\} \xrightarrow{\ \sim\ } X.$$

Let's assume this fact first and proceed to prove $(C)_m$.

Let $X^0 \subset M^{m-1}$ be the set of a s.t. X is uniformly étale on some neighb. of a.

<u>Claim</u>. X^0 is dense in M^{m-1}.

Assuming further this claim, let $\mathcal{D}$ be a c.d. of M^{m-1} s.t. X^0 is $\mathcal{D}$-adapted. If C is a cell, we distinguish two cases:

(1) C is <u>not</u> open. Then (see end of Section 3) we have a definable homeomorphism $C \xrightarrow{p_c} \tilde{C}$ with $\tilde{C}$ open (in $M^{\Sigma c_i}$ if $C \in \mathrm{Cells}_c(M)$); by induction $(C)_{<m}$, $X_C \longrightarrow C$ is uniformly finite (it is isomorphic to $(p_c^{-1})^* X_c \longrightarrow \tilde{C}$), say

$$\sup_{a \in C} |X_a| = k_C .$$

(2) C is <u>open</u>: then $C \cap X^0 \neq \emptyset$ so $C \subset X^0$; thus $X_C \longrightarrow C$ is locally unif^{ly} étale,

so it is trivial by the proposition, in particular unif^{ly} finite, say $\sup_{a \in C} |X_a| = k_C$.

So we conclude that

$$\forall a \in M^{m-1}, \quad |X_a| \leq \sup_{C \in \mathcal{D}} k_C < +\infty.$$

We now prove the claim.

Let $V \subset M^{m-2}$ be d-connected, $I \subset M$ open interval. We need to find an $a \in X^\circ$ in $V \times I$.

For any $x \in V$, $X_{\{x\} \times I} \longrightarrow I$ is finite, so by the Uniform Finiteness Theorem (for subsets of M^2, cf. Chapter V) we find a cell decomposition of I (depending on x) so that $X_{\{x\} \times J} \longrightarrow J$ is unif^{ly} étale for all open intervals $J \subset I$.

So the set Y° of $(x,t) \in V \times I$ s.t. $X_{\{x\} \times I} \rightarrow I$ is unif^{ly} étale around (x,t) is dense in $V \times I$.

We then apply $(A)_{m-1}$ to get a c.d. $\mathcal{D}$ of M^{m-1} s.t. $V \times I$ and Y° are $\mathcal{D}$-adapted.

Any open cell C of $\mathcal{D}$ is contained in Y^0; it contains $V' \times I'$ for some $\begin{cases} d\text{-connected open set } V' \subset M^{m-2} \\ \neq \emptyset \text{ open interval } I' \subset M \end{cases}$ and since $X_{\{x\} \times I'}$ is locally unif^{ly} étale over I', it is trivial by the proposition. So

$$|X_{(x,t)}| = |X_{(x,s)}| \quad , \quad \begin{cases} x \in V' \\ s, t \text{ in } I' \end{cases}$$

Pick $t_0 \in I'$; by $(C)_{m-1}$, the space $X_{V' \times \{t_0\}}$ is unif^{ly} finite over V'.

Now for $(x, t) \in V' \times I'$, we get

$$|X_{(x,t)}| = |X_{(x,t_0)}| \leq N$$

i.e. $X_{V' \times I'}$ is unif^{ly} finite.

In particular, we find $C' \subset V' \times I'$ open definable and $k \in \mathbb{N}$ s.t. $|X_a| = k$ for all $a \in C'$. Define

$$f_1, \ldots, f_k : C' \longrightarrow M$$

so that $\quad f_1(a) < \ldots < f_k(a)$

and $\qquad X_a = \{(a, f_i(a)) \mid 1 \leq i \leq k\}$

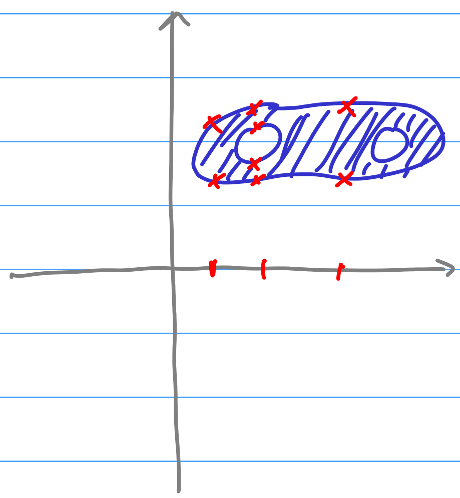
for $a \in C'$. Apply $(B)_{m-1,k}$ to get each f_i continuous on some open cell of a c.d. of C'; then over this cell, X is unif^{ly} étale and the cell is contained in $(V \times I) \cap X^\circ$.

This establishes the claim, and Step 2 (modulo the topological proposition). $\square$

<u>Proof of Step 3</u> - [$(A)_{<m}$, $(B)_{<m,k}$, $(C)_m \Rightarrow (A)_m$]

Let Y be the union of the finitely many sets we want to get adapted.

Let $X \subset M^m$ be the set of (a,b) such that $b \in (\overline{X}_a - \mathring{X}_a)$ [boundary of the fiber, in M]



By o-minimality, $X \longrightarrow M^{m-1}$ is <u>finite</u>. By $(C)_m$, X is unif^{ly} finite. Let then

$$N = \sup_{a \in M^{m-1}} |X_a|.$$

Now let : $\begin{cases} \tilde{D}_n = \{a \mid |X_a| = n\} \\ \tilde{D}_0 = M^{m-1} \end{cases}$

and define $f_0(a) = -\infty, \quad a \in \tilde{D}_0$

$f_i(a) = i\text{-th element } b \text{ s.t. } (a,b) \in X_a$
$(1 \leq i \leq n, \quad a \in \tilde{D}_n)$

and moreover $f_{n+1}(a) = +\infty$ if $a \in \tilde{D}_n$.

Then for a set Y to adapt, $m \leq n \leq N$, let

$$A(Y,n,m) = \{a \in \tilde{D}_n \mid \{a\} \times]f_m(a), f_{m+1}(a)[\subset Y\}$$

$$B(Y,n,m) = \{a \in \tilde{D}_n \mid (a, f_m(a)) \in Y\}.$$

Now by $(A)_{m-1}$, find a c.d. $\tilde{\mathcal{D}}$ of M^{m-1} s.t. all the $\tilde{D}_n$, all $A(Y,n,m)$, all $B(Y,n,m)$ are adapted. Further refine this using $(B)_{m-1}$ to find $\tilde{\tilde{\mathcal{D}}}$ s.t. all f_i's are continuous on the cells of $\tilde{\tilde{\mathcal{D}}}$ contained in their domain of definition.

And (finally) define $\mathcal{D}$ to be the family of cells $\begin{cases} \Gamma(f_m|C), & 1 \leq m \leq N \\ & C \subset \tilde{D}_m \text{ cell of } \tilde{\mathcal{D}} \\ ]f_m|C, f_{m+1}|C[, & 0 \leq m \leq N \\ & C \subset \tilde{D}_m \text{ cell of } \tilde{\mathcal{D}} \end{cases}$

Check (!) then that this gives the desired c.d. of M^n ...

$\square$

<u>Proof of Step 4.</u> $[\,(A)_{\leq m}, (B)_{<m} \Rightarrow (B)_{m,1}\,]$

We will need a lemma to prove continuity:

<u>Lemma.</u> Y topological space, X_1, X_2 totally ordered sets with order topology, $f : Y \times X_1 \longrightarrow X_2$.

If (1) $\forall y$, $x \longmapsto f(y,x)$ is continuous and monotone on X_1

(2) $\forall x$, $y \longmapsto f(y,x)$ is continuous

Then f is continuous.

Let $f : \underset{M^m}{\overset{\cap}{X}} \longrightarrow M$ be definable. It suffices to find a finite partition of X in def. subsets on which f is continuous, since we can then use $(A)_m$ to get a c.d. of X for which each of these subsets is adapted, and then f is continuous on each cell. Moreover, by a similar argument, we may assume that X is a cell (using a c.d. of X).

If X is not an open cell, using the definable homeomorphism $p_c : X \longrightarrow \tilde{X}$, $\tilde{X}$ open cell in some M^n, $n < m$, we obtain the result by induction using $(B)_{n,1}$.

So assume X is open.

We define $X^0 \subset X$ as the set of $(a,b) \in M^{m-1} \times M$ such that:

$\exists$ U open neigh of a, I open interval containing b with

$$\begin{cases} (1) & U \times I \subset X \\ (2) & \forall x \in U, \quad t \longmapsto f(x,t) \text{ is continuous} \\ & \qquad\qquad\qquad\qquad\quad + \text{monotone on } I \\ (3) & \begin{cases} U \longrightarrow M \\ x \longmapsto f(x,b) \end{cases} \text{ is continuous at } b \end{cases}$$

Note that $X^0 \subset X$ is definable.

Claim - X^0 is dense in X.

We assume this; let then $\mathcal{D}$ be a c.d. of M^m s.t. X, X^0 are adapted. If C is a cell of $\mathcal{D}$ then:

(1) if C is not open, we are done by induction as before

(2) if C is open, then $C \subset X^0$, and by construction any $a \in C$ is contained in some $U \times I$ s.t. $f|U \times I : U \times I \longrightarrow M$ satisfies the assumptions of the continuity lemma. So $f|C$ is continuous.

Now to prove the claim...

Consider $V \subset M^{m-1}$ open $\neq \emptyset$, $J \subset M$ open interval, such that $V \times J \subset X$. We need to show that $(V \times J) \cap X^{\circ} \neq \emptyset$.

Define $s : V \longrightarrow M$

s.t. $s(x) = \max \{ y \in]\inf J, \sup J] \ / \ f|\{x\} \times]\inf J, y[$ is continuous + monotone$\}$

which is well-defined by the local structure theorem.

Then $s : V \longrightarrow M$ is definable; by $(B)_{m-1,1}$, we find a c.d. of V on which cells s is continuous. Let W be an open cell of this c.d. and $x_0 \in W$. By continuity there is a $b > \inf(J)$ and $W' \subset W$ open definable with

$$s(x) \geq b \quad , \quad x \in W'.$$

Let $y_0 \in]\inf J, b[$. The map $x \longmapsto f(x, y_0)$ is definable on W', so by $(B)_{m-1}$, we find

$W'' \subset W'$ definable open s.t. $x \longmapsto f(x, y_0)$ is definable on W''.

But then $W'' \times \{y_0\} \subset X^\circ$, hence the claim.

This concludes the proof of the Decomposition Theorem, up to the two topological facts.

<u>Proof of the Proposition.</u> We have $X \longrightarrow B$ loc$^{\text{lly}}$ uniformly étale.

Let $a \in B$ and V an open $\overset{d\text{-connected}}{\vee}$ neighb. of a s.t. $X_V \longrightarrow V$ is unif$^{\text{ly}}$ étale. Let $k = |X_a|$ and

$$b_1 < \dots < b_k$$

the elements s.t. $X_a = \{(a, b_i)\}$. By definition, there are $f_i : V \longrightarrow M$ continuous definable s.t. $b_i = f_i(a)$.

Let $1 \leq i < j \leq k$. Note that $V = V_> \cup V_< \cup V_=$ with

$$V_\square = \{ x \in V \mid f_i(x) \,\square\, f_j(x) \}.$$

By continuity, $V_<$ and $V_>$ are open; by the étale property, so is $V_=$ (standard topological property: write $(U \times I) \cap X = \Gamma(f)$; then $f = f_i = f_j$ on U); since V is d-connected, and $a \in V_<$, this means that $V = V_<$, so $f_i(x) < f_j(x)$ on V.

Let now $(x, y) \in X_V$. By def. we get an

$$f : V \longrightarrow M$$

continuous definable with $f(x) = y$. There is some i s.t. $f_i(a) = f(a)$, and as before this holds on some open set in V, hence $f = f_i$ on V.

We deduce that $X_x = \{ (x, f_i(x)) \}$ for all $x \in V$, so $X_V \longrightarrow V$ is trivial:

$$\begin{cases} V \times \{1, \dots, k\} \xrightarrow{\ \sim\ } X_V \\ (x, i) \longmapsto f_i(x). \end{cases}$$

All this shows that $a \longmapsto |X_a|$ is locally constant, on B, hence constant by d-connectedness. Then the f_i are continuous and the above formula is a global trivialization.

$\square$